Contents

Chapter 10

Optimum Linear Systems

Design of an optimum system: Necessary informations on hand:

(1) **Input specification:**
random, deterministic, correlation function etc.

(2) **System constraints:**
linear, non-linear, time invariant etc.

(3) **Criterion of optimality:** *meaningful measure of goodness*
(e.g.)

(i) Minimization of error(MSE): $\longrightarrow$ Wiener filter
(ii) Maximization of SNR: $\longrightarrow$ Matched filter

$\vdots$

10.1 Systems maximizing SNR: *Matched filter*

Objective:

Designing an LTI system (or filter) that maximizes the output signal to noise ration (SNR) at $t = t_0$.

Problem statement: digital communication (example)

Figure 10.1: Input and output signals in digital communication system.

Goal:
At the receiver, we want to design an optimum system in order to determine whether the signal sent was "1" or "0":

Figure 10.2: Concept of an optimum receiver.

where $x(t)$ ia a (sum of) deterministic signal.

Informations on hand:

(1) $x(t)$ is a deterministic signal, and the noise $n(t)$ is a stationary random signal with its PSD of $S_{NN}(\omega)$.

(2) The receiver is an LTI system.

(3) Criterion of optimality: maximize the output SNR at $t = t_0$

$$\underset{h(t),H(\omega)}{argmax} \left(\frac{\hat{S}_o}{N_o} \right) \triangleq \underset{h(t),H(\omega)}{argmax} \frac{|x_o(t_0)|^2}{E\left[N_o^2(t)\right]} \tag{10.1}$$

where [1]

$$\hat{S}_o \triangleq |x_o(t_0)|^2 \quad : \quad \text{output signal power at } t = t_0$$

$$N_o \triangleq E\left[N_o^2(t)\right] \quad : \quad \text{output average noise power}$$

[1] Since the noise is random, we must rely on the statistical (or average) power of the noise rather than the noise power at $t = t_0$.

10.1.1 Matched filter for colored noise

Figure 10.3: An LTI optimum receiver.

(i) The signal power of the output at $t = t_0$:

$$\begin{aligned} x_o(t) &= h(t) * x(t) \quad \text{(by linearity of system)} \\ &\overset{\text{or}}{=} \mathcal{F}^{-1}\{H(\omega)X(\omega)\} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t}d\omega \end{aligned}$$

Therefore, we have the output at $t = t_0$ as:

$$x_o(t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t_0}d\omega \tag{10.2}$$

(ii) The average noise power of the output:

$$N_o \triangleq E\left[N_o^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left|H(\omega)\right|^2 d\omega \tag{10.3}$$

where $\left|H(\omega)\right|^2$ is the power transfer function of the system, and $S_{NN}(\omega)\left|H(\omega)\right|^2$ corresponds to the PSD $S_{N_oN_o}(\omega)$ of the output noise.

Applying (10.2) and (10.3) to (10.1), the optimal system which maximizes the output SNR is the $H(\omega)$ satisfying the following:

$$\underset{H(\omega)}{argmax} \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left|H(\omega)\right|^2 d\omega} \triangleq \underset{H(\omega)}{argmax}\, F \tag{10.4}$$

$\Longrightarrow$ To find $H_{opt}(\omega)$ satisfying (10.4), we apply the Schwarz inequality.

Inner product between two signals:

Definition 10.1 The inner (or scalar) product $< x(t), y(t) >$ b/w two signals[2] $x(t)$ and $y(t)$ can be any functions of $x(t)$ & $y(t)$ providing a *scalar*(complex) satisfying the following three requirements:

(i) Linearity: $< \alpha x(t) + \beta y(t), z(t) >= \alpha < x(t), z(t) > +\beta < x(t), z(t) >$

(ii) Symmetry: $< x(t), y(t) >=< y(t), x(t) >^*$

(iii) Non degeneracy: $< x(t), x(t) > \triangleq ||x(t)||^2 \geq 0 \quad \& \quad ||x(t)||^2 = 0$ iff $x(t) = 0$

(cf) We call $||x(t)||$, the *norm* of $x(t)$.

Schwarz inequality:

Theorem 10.1 The inner product of two signals and their norms should satisfy the following inequality:

$$|< x(t), y(t) >| \leq ||x(t)|| \cdot ||y(t)|| \tag{10.5}$$

with equality if (i) either $x(t)$ or $y(t)$ is zero, or (ii) $x(t) = \alpha y(t)$ where α is a (complex) scalar.

proof:

Let $x(t) = x$ and $y(t) = y$ for notational convenience, and consider a *non-negative* quantity $||x + \alpha y||^2$: [3]

$$\begin{aligned}
||x + \alpha y||^2 &\triangleq < x + \alpha y, x + \alpha y > \\
&= < x, x > + < x, \alpha y > + < \alpha y, x > + < \alpha y, \alpha y > \\
&= < x, x > +\alpha^* < x, y > +\alpha < y, x > +|\alpha|^2 < y, y > \quad \text{(linearity)} \\
&= ||x||^2 + \alpha^* < x, y > +\alpha < x, y >^* +|\alpha|^2 ||y||^2 \quad \text{(symmetry)}
\end{aligned}$$

Since α can be arbitrary, we **choose**:

$$\alpha = -\frac{< x, y >}{||y||^2}$$

[2]Complex signals in general.
[3]Note that $< x, \alpha y >=< \alpha y, x >^*= (\alpha < y, x >)^* = \alpha^* < y, x >^*= \alpha^* < x, y >$.

Then,

$$\begin{aligned}
||x+\alpha y||^2 &= ||x||^2 - \frac{<x,y>^*}{||y||^2} <x,y> - \frac{<x,y>}{||y||^2} <x,y>^* + \frac{|<x,y>|^2}{||y||^4}||y||^2 \\
&= ||x||^2 - \frac{|<x,y>|^2}{||y||^2} \\
&> 0 \quad \text{(should be)}
\end{aligned}$$

From which we get:

$$|<x,y>|^2 \leq ||x||^2 \cdot ||y||^2$$

And therefore:

$$|<x,y>| \leq ||x|| \cdot ||y||$$

Also, since $||x+\alpha y||^2 = 0$ if and only if $x + \alpha y = 0$ from the non-degeneray property of the inner product, the equality holds iff $x = -\alpha y$.

q.e.d.

Example 10.1

Define an inner product b/w $x(t)$ and $y(t)$ as follows:

$$<x(t), y(t)> \triangleq \int_{-\infty}^{\infty} x(t)y^*(t)dt \tag{10.6}$$

Then, prove that (10.6) satisfies all of the three requirement for an inner product, and thus (10.6) could be a valid inner product.

Solution: : assignment

Using the definition of the inner product as in (10.6) and applying the Schwarz inequality [4] of (10.5) by substituting:

$$\begin{cases} \text{(i) } x \longrightarrow \sqrt{S_{NN}(\omega)}H(\omega) \\ \\ \text{(ii) } y \longrightarrow \dfrac{X^*(\omega)e^{-j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}} \end{cases}$$

we have:

$$\left| \int_{-\infty}^{\infty} \sqrt{S_{NN}(\omega)}H(\omega) \cdot \frac{X(\omega)e^{j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}} d\omega \right|^2$$

$$\leq \int_{-\infty}^{\infty} \sqrt{S_{NN}(\omega)}H(\omega)\sqrt{S_{NN}(\omega)}H^*(\omega)d\omega \cdot \int_{-\infty}^{\infty} \frac{X^*(\omega)e^{-j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}} \cdot \frac{X(\omega)e^{j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}} d\omega$$

which reduces to:

$$\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t_0}d\omega \right|^2 \leq \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} S_{NN}(\omega) \left|H(\omega)\right|^2 d\omega \cdot \int_{-\infty}^{\infty} \frac{\left|X(\omega)\right|^2}{S_{NN}(\omega)} d\omega$$

Therefore, (10.4) becomes:

$$F \triangleq \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)X(\omega)e^{j\omega t_0}d\omega \right|^2}{\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{NN}(\omega) \left|H(\omega)\right|^2 d\omega} \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|X(\omega)\right|^2}{S_{NN}(\omega)} d\omega$$

Notice that the maximum SNR occurs as follows:

$$\max F = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\left|X(\omega)\right|^2}{S_{NN}(\omega)} d\omega$$

when

$$\sqrt{S_{NN}(\omega)}H(\omega) = \alpha \frac{X^*(\omega)e^{-j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}}$$

[4] Schwarz inequality: $\left| \int_{-\infty}^{\infty} xy^* d\omega \right|^2 \leq \int_{-\infty}^{\infty} xx^* d\omega \int_{-\infty}^{\infty} yy^* d\omega$.

Therefore, the transfer function of the optimal system which maximizes the output SNR at $t = t_0$ is as follows:

$$H_{opt}(\omega) = \alpha \cdot \frac{X^*(\omega)e^{-j\omega t_0}}{2\pi\sqrt{S_{NN}(\omega)}}$$

Note:

(i) $H_{opt}(\omega)$ is proportional to $X^*(\omega)$, in other words *matched to unput* (i.e. depends on the input $x(t)$), and this is why it is called a **matched filter**.

(ii) $H_{opt}(\omega)$ could have arbitrary gain via α, but α affects both the signals and the noise, thus has no effect on the SNR.

(iii) The choice of t_0 only affects the *delay* of the output signal, and we must choose appropriate t_0 in order to make the system *causal* [5].

10.1.2 Matched filter for white noise

Suppose $S_{NN}(\omega) = \frac{N_0}{2}$, i.e. the noise is *white*, then the optimal filter becomes:

$$H_{opt}(\omega) = \alpha \cdot \frac{X^*(\omega)e^{-j\omega t_0}}{2\pi\,(0.5N_0)} \overset{\text{let}}{=} \beta X^*(\omega)e^{-j\omega t_0} \qquad \text{where } \beta = \tfrac{\alpha}{\pi N_0}$$

The impulse response $h_{opt}(t)$ then becomes;

$$\begin{aligned}
h_{opt}(t) = \mathcal{F}^{-1}\left\{H_{opt}(\omega)\right\} &= \beta \cdot \mathcal{F}^{-1}\left\{X^*(\omega)e^{-j\omega t_0}\right\} \\
&= \beta\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t_0}e^{j\omega t}d\omega \\
&= \beta\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega(t_0-t)}d\omega\right]^* \\
&= \beta x^*(t_0 - t) \\
&= \beta x(t_0 - t) \quad : \text{if } x(t) \text{ is real} \\
& \qquad : \text{matched filter (expressed in terms of } x(t))
\end{aligned}$$

[5] $h_{opt}(t) = \mathcal{F}^{-1}\left\{H_{opt}(\omega)\right\}$

Figure 10.4: The impulse response of a matched filter under white noise.

10.2 Systems minimizing MSE: *Wiener filter*

Objective:

Designing a system which provides a *good estimate* of the future, present, and the past value of the input signal.

Wiener filter:

(i) **prediction** : future

(ii) **filtering** : present

(iii) **smoothing** : past

Figure 10.5: Concept of a Wiener filter.

Informations on hand (assumptions) [6]:

(1) $X(t)$ and $N(t)$ are JWSS, and $E[N(t)] = 0$.

(2) The system is LTI, and $h(t)$ is real.

(3) Criterion of optimality: *minimize the MSE*

$$\underset{H(\omega)}{argmin}\, E\left[\{X(t+t_0) - Y(t)\}^2\right] \tag{10.7}$$

where $\epsilon^2(t) \triangleq \{X(t+t_0) - Y(t)\}^2$ corrsponds to the *squared error.*

[6] These are the (1) input signal spec. (2) system spec. and (3) criterion of optimality respectively.

Now[7],

$$
\begin{aligned}
E\left[\epsilon^2(t)\right] &= E\left[\{X(t+t_0) - Y(t)\}^2\right] \\
&= E\left[X^2(t+t_0)\right] - 2E\left[X(t+t_0)Y(t)\right] + E\left[Y^2(t)\right] \\
&= R_{XX}(0) - 2R_{XY}(-t_0) + R_{YY}(0) \\
&= R_{XX}(0) - 2R_{YX}(t_0) + R_{YY}(0) \qquad (10.8)
\end{aligned}
$$

Here, each term in (10.8) is as follows:

(i) Input power:

$$
\begin{aligned}
R_{XX}(0) &= \mathcal{F}^{-1}\left\{S_{XX}(\omega)\right\}_{\tau=0} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega
\end{aligned}
$$

(ii) Output power:

$$
\begin{aligned}
R_{YY}(0) &= \mathcal{F}^{-1}\left\{S_{YY}(\omega)\right\}_{\tau=0} \\
&= \mathcal{F}^{-1}\left\{|H(\omega)|^2 S_{WW}(\omega)\right\}_{\tau=0} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{WW}(\omega)d\omega
\end{aligned}
$$

[7] Be reminded that $X(t)$ and $Y(t)$ are JWSS.

(iii) Cross correlation at $t = t_0$[8]:

$$
\begin{aligned}
R_{YX}(t_0) &= E\left[Y(t)X(t+t_0)\right] \\
&= E\left[X(t+t_0)\int_{-\infty}^{\infty} h(\tau)W(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} h(\tau)R_{XW}(-\tau-t_0))d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)R_{WX}(\tau+t_0))d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WX}(\omega)e^{j\omega(\tau+t_0)}d\omega d\tau \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WX}(\omega)\left\{\int_{-\infty}^{\infty} h(\tau)e^{j\omega\tau}d\tau\right\}e^{j\omega t_0}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WX}(\omega)H^*(\omega)e^{j\omega t_0}d\omega \quad \text{(since } h(t) \text{ is real)}
\end{aligned}
$$

Applying (i), (ii), and (iii) to (10.8), we get:

$$
E\left[\epsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) - 2S_{WX}(\omega)H^*(\omega)e^{j\omega t_0} + |H(\omega)|^2 S_{WW}(\omega)\right\}d\omega \quad (10.9)
$$

Note: The integrand in (10.9):

The first and the third terms (i.e. $S_{XX}(\omega)$ and $|H(\omega)|^2 S_{WW}(\omega)$) are *real* and thus have *maginitude only*, while the second term $S_{WX}(\omega)H^*(\omega)e^{j\omega t_0}$ is *complex* and has both the *magnitude* and *phase*.

Express:

$$
\begin{cases}
\text{(i) } H(\omega) = |H(\omega)|\, e^{j\Phi_H(\omega)} \\
\text{(ii) } S_{WX}(\omega) = M(\omega)e^{j\Theta(\omega)}
\end{cases} \quad (10.10)
$$

then, (10.9) becomes:

$$
\begin{aligned}
E\left[\epsilon^2(t)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) + |H(\omega)|^2 S_{WW}(\omega)\right\}d\omega \\
&\quad -\frac{1}{2\pi}\int_{-\infty}^{\infty} 2M(\omega)\cdot|H(\omega)|\, e^{j(\omega t_0 - \Phi_H(\omega) + \Theta(\omega))}d\omega \quad (10.11)
\end{aligned}
$$

[8] Also be reminded that $X(t)$ and $N(t)$ are JWSS.

To find $H(\omega) \ni: argmin_{H(\omega)} E\left[\epsilon^2(t)\right]$, we follow the steps below:

(1) Find $\Phi_H(\omega)$ maximizing the second ontegral in (10.11).

(2) Find $|H(\omega)|$ minimizing (10.11) with the optimum $\Phi_H(\omega)$

(1) **Optimum phase $\Phi_H(\omega)$:**

To maximize the second integral in (10.11), we choose:

$$\omega t_0 - \Phi_H(\omega) + \Theta(\omega) = 0$$

which gives us:

$$\Phi_H(\omega) = \omega t_0 + \Theta(\omega) \tag{10.12}$$

(2) **Optimum magnitude $|H(\omega)|$:**

Substituting (10.12) into (10.11), we get:

$$E\left[\epsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) - 2M(\omega)\cdot|H(\omega)| + |H(\omega)|^2 S_{WW}(\omega)\right\}d\omega$$

Completing the square of the integrand in $|H(\omega)|$, we have:

$$\begin{aligned}
&E\left[\epsilon^2(t)\right] \\
= \;& \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{WW}(\omega)\left\{|H(\omega)|^2 - 2\frac{M(\omega)}{S_{WW}(\omega)}|H(\omega)| + \frac{M^2(\omega)}{S_{WW}^2(\omega)} - \frac{M^2(\omega)}{S_{WW}^2(\omega)}\right\}\right. \\
&\left. + S_{XX}(\omega)\right]d\omega \\
= \;& \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{WW}(\omega)\left(|H(\omega)| - \frac{M(\omega)}{S_{WW}(\omega)}\right)^2 - \frac{M^2(\omega)}{S_{WW}^2(\omega)} + S_{XX}(\omega)\right]d\omega
\end{aligned} \tag{10.13}$$

The $|H(\omega)|$ which minimizes (10.13) is obviously [9],

$$
\begin{aligned}
|H_{opt}(\omega)| &= \frac{M(\omega)}{S_{WW}(\omega)} \\
&= \frac{S_{WX}(\omega)e^{-j\Theta(\omega)}}{S_{WW}(\omega)}
\end{aligned}
\tag{10.14}
$$

Combining (10.12) and (10.14), we get:

$$
\begin{aligned}
H_{opt}(\omega) &= |H_{opt}(\omega)| \cdot e^{j\Phi_H(\omega)} \\
&= \frac{S_{WX}(\omega)}{S_{WW}(\omega)} e^{-j\Phi_H(\omega)} \cdot e^{j\omega t_0 + j\Theta(\omega)} \\
&= \frac{S_{WX}(\omega)}{S_{WW}(\omega)} e^{j\omega t_0}
\end{aligned}
$$

: trasfer function of the **Wiener filter**

Special case: $W(t) = X(t) + N(t)$

If $X(T)$ and $N(t)$ are *uncorrelated* in addition to JWSS, then [10] [11]

(i) The PSD of $W(t)$:

$$
\begin{aligned}
S_{WW}(\omega) &= S_{XX}(\omega) + S_{XN}(\omega) + S_{NX}(\omega) + S_{NN}(\omega) \\
&= S_{XX}(\omega) + S_{NN}(\omega)
\end{aligned}
$$

(ii) Cross PSD of $W(t)$ & $X(t)$:

$$S_{WX}(\omega) = S_{XX}(\omega) + S_{NX}(\omega) = S_{XX}(\omega)$$

Therefore, the Wiener filter becomes:

$$H_{opt}(\omega) = \frac{S_{XX}(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)} e^{j\omega t_0}$$

[9] Recall from (10.10) that $S_{WX}(\omega) = M(\omega)e^{j\Theta(\omega)}$, and thus $M(\omega) = S_{WX}(\omega)e^{-j\Theta(\omega)}$.

[10] If $X(t)$ and $N(t)$ are uncorrelated, then $S_{XN}(\omega) = S_{NX}(\omega) = 2\pi\overline{X}\overline{N}\delta(\omega)$, where in this case $\overline{X} = \overline{Y} = 0$, and thus $S_{XN}(\omega) = S_{NX}(\omega) = 0$.

[11] Also, notice that since $X(t)$ and $N(t)$ are uncorrelated, $R_{XN}(\tau) = E[X(t)]E[N(t)] = 0$ since $E[N(t)] = 0$.

The minimum MSE:

From (10.11), the resulting minimum MSE is:

$$E\left[\epsilon^2(t)\right]_{min} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) - \frac{M^2(\omega)}{S_{WW}(\omega)}\right\}d\omega$$

where $M(\omega) = |S_{WX}(\omega)|$.

(cf)
If $X(T)$ and $N(t)$ are *uncorrelated*, $S_{WX}(\omega) = S_{XX}(\omega)$ and thus the corresponding minimum MSE becomes as follows:

$$\begin{aligned} E\left[\epsilon^2(t)\right]_{min} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{XX}^2(\omega) + S_{XX}(\omega)S_{NN}(\omega) - S_{XX}^2(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{XX}(\omega)S_{NN}(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)}d\omega \end{aligned}$$