Contents

Chapter 4

The Operations on One Random Variable - Expectation

4.1 Mathematical expectation

Definition 4.1 Given a probability space $(S, \mathcal{F}, P)$ and a random variable $X(\omega)$, the mathematical expectation of $X(\omega)$ is defines as:

$$E[X(\omega)] \stackrel{d}{=} E[X] \triangleq \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

(cf) Other terms:

(i) Mean

(ii) Average

(iii) Statistical average

Note:
The above definition is good(valid) for all types of random variables, whether it is *continuous*, *discrete*, or *mixed* r.v.'s.

Remarks:

(1) Recall that if $X(\omega)$ is a discrete type r.v., then the p.d.f. of $X(\omega)$ can be written as:

$$f_X(x) = \sum_{i=1}^{N} p(x_i)\delta(x - x_i)$$

Therefore, the mathematical expectation of a discrete r.v. $X(\omega)$ becomes:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x)dx \\
&= \int_{-\infty}^{\infty} x \sum_{i=1}^{N} p(x_i)\delta(x - x_i)dx \\
&= \sum_{i=1}^{N} p(x_i) \int_{-\infty}^{\infty} x\delta(x - x_i)dx \\
&= \sum_{i=1}^{N} x_i \cdot p(x_i) \quad \text{(by sifting property of } \delta(x).) \\
&: \quad \text{well known form of expectation for discrete r.v..}
\end{aligned}$$

Example 4.1

In a game of tossing a coin, suppse that you get 100 won if head comes up, while you lose 100 won if tail shows up, from which the following random variable $X(\omega)$ is defines:

$$X(H) = 100 \stackrel{d}{=} x_1$$

$$X(T) = -100 \stackrel{d}{=} x_2$$

Assuming that the coin is fair, what is the expected value of $X(\omega)$?

Figure 4.1: The sample space S of coin tossing game.

Solution:
Since the coin is fair, we have:

$$P(H) = P(T) = \frac{1}{2}$$

Figure 4.2: The p.d.f. of $X(\omega)$.

Therefore, the mathematical expectation of X is then:

$$E[X] = \sum_{i=1}^{2} x_i \cdot p(x_i) = 100 \times \frac{1}{2} + (-100) \times \frac{1}{2} = 0$$

which means that after many trials (of the coin tossing game), you would not gain or lose any money at all, just wasting your valuable time!!!

Example 4.2

In the same game of tossing a coin, suppose that you can manipulate the coin such that:

$$P(H) = p, \qquad \text{and} \qquad P(T) = 1 - p$$

Assume tha you get 100 won if head comes up, while you lose 200 won if tail shows up, which might be a sweet attraction to other people to join the game. If you do not want to lose your money, how should you manipulate the coin for the value of p?

Solution:
From the rule of the game, we have:

$$X(H) = 100$$

$$X(T) = -200$$

Therefore, the mathematical expectation of X must then be:

$$E[X] = 100p + (-200)(1 - p) = 300p - 200 \geq 0 \text{ (should be)}$$

which gives you the coin must be made biased so that $p \geq \frac{2}{3}$.

(2) If the p.d.f. $f_X(x)$ is an *even* function (i.e. symmetric about $x = 0$), then;

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx = 0$$

proof:

$$\begin{aligned} E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx &= \int_{-\infty}^{0} x \cdot f_X(x)dx + \int_{0}^{\infty} x \cdot f_X(x)dx \\ &\quad \text{(let } u = -x \text{ in the 1st integral)} \\ &= \int_{\infty}^{0} -u \cdot f_X(-u)(-du) + \int_{0}^{\infty} x \cdot f_X(x)dx \\ &= -\int_{0}^{\infty} x \cdot f_X(x)dx + \int_{0}^{\infty} x \cdot f_X(x)dx \\ &= 0 \end{aligned}$$

q.e.d.

Figure 4.3: A symmetric p.d.f. $f_X(x)$.

(cf.)

(i) Notice that since $f_X(x)$ is an even function, while x is an odd function, the integrand in the mathematical expectation becomes odd, of which integration from $-\infty$ to ∞ therefore is zero.

$$E[X] = \int_{-\infty}^{\infty} \underbrace{x}_{\text{odd}} \cdot \underbrace{f_X(x)}_{\text{even}} dx$$

(ii) The above fact applies to the example of coin tossing game, where the p.d.f. is an even function.

(3) Generalization of (2):

If the p.d.f. $f_X(x)$ is symmetric about $x = a$, i.e. $f_X(x+a) = f_X(-x+a)$, then;

$$E[X] = a$$

Figure 4.4: A p.d.f. such that $f_X(x+a) = f_X(-x+a)$.

proof:

We will consider $E[X-a]$, and show that $E[X-a] = 0$, which in turn will indicate that $E[X] = a$, since:

$$\begin{aligned}
E[X-a] &= \int_{-\infty}^{\infty} (x-a) f_X(x) dx \\
&= \int_{-\infty}^{\infty} x f_X(x) dx - \int_{-\infty}^{\infty} a f_X(x) dx \\
&= \int_{-\infty}^{\infty} x f_X(x) dx - a \int_{-\infty}^{\infty} f_X(x) dx \\
&= E[X] - a \\
&\equiv 0
\end{aligned}$$

which means that:

$$E[X] = a$$

Now, we show that $E[X-a] = 0$ in the following:

$$
\begin{aligned}
E[X-a] &= \int_{-\infty}^{\infty}(x-a)f_X(x)dx \\
&\quad \text{(let } x-a=y\text{)} \\
&= \int_{-\infty}^{\infty} y\cdot f_X(y+a)dy \\
&= \int_{-\infty}^{0} y\cdot f_X(y+a)dy + \int_{0}^{\infty} y\cdot f_X(y+a)dy \\
&= \text{(let } z=-y \text{ in the 1st integral)} \\
&= \int_{\infty}^{0} -z\cdot f_X(-z+a)(-dz) + \int_{0}^{\infty} y\cdot f_X(y+a)dy \\
&= -\int_{0}^{\infty} z\cdot f_X(-z+a)dz + \int_{0}^{\infty} y\cdot f_X(y+a)dy \\
&= 0 \quad \text{(since } f_X(-y+a)=f_X(y+a)\text{)}
\end{aligned}
$$

Conditional expectation:

Definition 4.2 The conditional expectation of a r.v. $X(\omega)$ given an event B is denoted and defined as follows:

$$
E[X|B] \triangleq \int_{-\infty}^{\infty} x\cdot f_X(x|B)dx
$$

4.2 Expectation of functions of a random variable

Theorem 4.1 Given a transformation (or function) $y = g(x)$, we define a new rnadom variable $Y \ni$:

$$Y = g(X)$$

Then, the mathematical expectation $E[Y]$ of the newly defined r.v. Y is given by:

$$E[Y] = E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)dx$$

Proof: See Papoulis at p.105.

Remarks:

(1) The ordinary and straightforward way of calculatin $E[Y]$ would be as follows:

  (i) Compute $f_Y(y)$. [1]

  (ii) The, calculate $E[Y]$ according to the definition of the mathematical expectation:

$$E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)dy$$

(2) This applies to the case when we proved $E[X] = a$ if the p.d.f. $F_X(x)$ is symmetric around $x = a$, by showing $E[X - a] = 0$.

[1] We will later discuss the methodology of how we get the p.d.f. of a newly defined r.v. Y which is a function of a r.v. X, i.e. $Y = g(X)$.

1. Moments (m_n):

Definition 4.3 The n-th moment of a r.v. $X(\omega)$ is defined as:

$$m_n = n\text{-th moment} \triangleq E\left[X^n\right] = \int_{-\infty}^{\infty} x^n \cdot f_X(x)dx$$

2. Central moments (μ_n):

Definition 4.4 The n-th central moment of a r.v. $X(\omega)$ is defined as:

$$\begin{aligned} \mu_n = n\text{-th central moment} \;\; &\triangleq \;\; E\left[(X - m_1)^n\right] \\ &= \;\; \int_{-\infty}^{\infty} (x - m_1)^n \cdot f_X(x)dx \end{aligned}$$

3. Variance (σ_X^2):

Definition 4.5 The variance of a r.v. $X(\omega)$ is defined as:

$$\sigma_X^2 = \text{variance} \triangleq E\left[(X - m_1)^2\right] = \mu_2 \quad \text{(2nd central moment)}$$

(cf.) Note that:

$$\sigma_X^2 = E[X^2] - m_1^2$$

4. Standard deviation (σ_X):

Definition 4.6 The standard deviation of a r.v. $X(\omega)$ is defined as the square root of the variance:

$$\sigma_X = \text{standard deviation} \triangleq \sqrt{\mu_2}$$

5. The characteristic function of a r.v. $X(\omega)$:

Definition 4.7 The characteristic function of a r.v. X is defined as:

$$\begin{aligned}\Phi(\omega) &\triangleq E\left[e^{j\omega X}\right] \\ &= \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)dx\end{aligned}$$

Note:

(i) Here, ω is just a *parameter(or variable)*, **not** a point in the sample space. Do not be confused!!!

(ii) Notice that the characteristic function in in the form of the *inverse Fourier transform* of $f_X(x)$.

(iii) Using the similarity to the Fourier transform pair, we can compute the p.d.f. $f_X(x)$ from $\Phi(\omega)$ as:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega)e^{-j\omega x}d\omega$$

(Cf.) The Leibnitz Rule:

Let

$$g(x) = \int_{\alpha(x)}^{\beta(x)} f(x,u)du$$

where $f(x,u)$ is a *continuous* function w.r.t. x and u, then the derivative of $g(x)$ can be expressed in the following form:

$$\frac{dg(x)}{dx} = f\left(x,\beta(x)\right)\frac{d\beta(x)}{dx} - f\left(x,\alpha(x)\right)\frac{d\alpha(x)}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} f(x,u)du$$

NOW, notice that:

$$\begin{aligned}
\frac{d}{d\omega}\Phi(\omega) &= \frac{d}{d\omega}\int_{-\infty}^{\infty} e^{j\omega x} f_X(x)dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial\omega}\left\{e^{j\omega x} f_X(x)\right\} dx \qquad \text{(by the Leibnitz rule)} \\
&= j\int_{-\infty}^{\infty} x e^{j\omega x} f_X(x)dx
\end{aligned}$$

Therefore, we have

$$\left.\frac{d}{d\omega}\Phi(\omega)\right|_{\omega=0} = j\int_{-\infty}^{\infty} x \cdot f_X(x)dx = jm_1$$

Likewise,

$$\left.\frac{d^2}{d\omega^2}\Phi(\omega)\right|_{\omega=0} = j^2\int_{-\infty}^{\infty} x^2 \cdot f_X(x)dx = j^2 m_2$$

$$\vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots$$

In general, we have:

$$\left.\frac{d^n}{d\omega^n}\Phi(\omega)\right|_{\omega=0} = j^n m_n \tag{4.1}$$

From (4.1), we can compute the **moments** of the r.v. X directly from the characteristic function in the following way:

$$m_n = \frac{1}{j^n} \cdot \left.\frac{d^n}{d\omega^n}\Phi(\omega)\right|_{\omega=0}$$

Moment generating function

$$M_X(\nu) \triangleq E\left[e^{\nu X}\right] = \int_{-\infty}^{\infty} e^{\nu x} f_X(x)dx$$

where nu is a real number.
In the same way as for the characteristic function, the n-th moment of r.v. X can be obtained via the moment generating function as follows:

$$m_n = \left.\frac{d^n}{d\nu^n}M_X(\nu)\right|_{\nu=0}$$

(cf.) Notice that the definition of $M_X(\nu)$ is similar to the *Laplace transform*.

4.3 Non-linear function of a random variable

[2]

Concept:

Given a non-linear system whose input is a r.v. $X(\omega)$ with its p.d.f. of $f_X(x)$, we wish to find the p.d.f. $f_Y(y)$ of the output r.v. $Y(\omega)$.

Figure 4.5: A Non-linear system with input X and the output Y.

(e.g.)

1. Square law device
2. Full wave rectifier
3. Saturated amplifier

Question:

Given a probability space $(S, \mathcal{F}, P)$, and a *continuous* r.v. $X(\omega): S \rightarrow R^1$-line for which $F_X(x)$ and $f_X(x)$ are known.
Define a new r.v. $Y(\omega) \ni: Y(\omega) = T\,[X(\omega)]$, where $y = T[x]$ is a *continuous* function of x ($T[\cdot]$ is called a transformation).

Figure 4.6: A non-linear transformation $T[\cdot]$.

Then, what is the distribution $F_Y(y)$ of the newly defined r.v. $Y(\omega)$, and/or corresponding p.d.f. $f_Y(y)$?

[2] The linear transformation can be considered as a special case.

Answer:

1. The distribution function: $F_Y(y)$

$$\begin{aligned} F_Y(y) &\overset{\triangle}{=} P\{\omega \,|\, Y(\omega) \le y\} \overset{\text{let}}{\equiv} P(Y \le y) \\ &= P\{\omega \,|\, T[X(\omega)] \le y\} \overset{\text{let}}{\equiv} P(T[X] \le y) \end{aligned} \tag{4.2}$$

2. The density function: $f_Y(y)$

$$f_Y(y) \overset{\triangle}{=} \frac{d}{dy} F_Y(y) \tag{4.3}$$

Note:
Notice that the cdf in (4.2) of $Y(\omega)$, which is in the form of a probability, can be evaluated via the cdf $F_X(x)$ or the p.d.f. $f_X(x)$ of $X(\omega)$, and thus the corresponding p.d.f. in (4.3) of $Y\omega)$ as well.

There exist three different cases: (simple case $\rightarrow$ general case)

1. $y = T[x]$ is a monotone increasing continuous function of x (i.e. $\frac{dy}{dx} > 0$):

Figure 4.7: A monotone increasing transformation $T[\cdot]$.

(1) The distribution function $F_Y(y)$:

In this case, we have:

$$P[Y \le y] = P[X \le x]$$

where $x = T^{-1}[y]$.

Therefore, the distribution function $F_Y(y)$ in (4.2) becomes:

$$\begin{aligned} F_Y(y) = P\{\omega \mid Y(\omega) \le y\} &= P\{\omega \mid X(\omega) \le x\} \\ &= F_X(x)\big|_{x=T^{-1}(y)} \end{aligned}$$

(2) The density function $f_Y(y)$:

Corrsponding p.d.f. $f_Y(y)$ is then:

$$\begin{aligned} f_Y(y) = \frac{dF_Y(y)}{dy} &= \frac{dF_X(x)}{dy} \\ &= \frac{dF_X(x)}{dx} \cdot \frac{dx}{dy} \\ &= f_X(x) \cdot \frac{1}{\frac{dy}{dx}} \\ &= f_X(x) \cdot \frac{1}{\left|\frac{dy}{dx}\right|} \quad (\text{since } \tfrac{dy}{dx} > 0) \end{aligned}$$

where $x = T^{-1}(y)$.

2. $y = T[x]$ is a monotone decreasing continuous function of x (i.e. $\frac{dy}{dx} < 0$):

Figure 4.8: A monotone decreasing transformation $T[\cdot]$.

(1) The distribution function $F_Y(y)$:

For the case of decreasing $T[\cdot]$, we have:

$$P[Y \leq y] = P[X \geq x] = 1 - P[X < x] = 1 - P[X \leq x]$$

where $x = T^{-1}[y]$.

cf: This is because $T[\cdot]$ is a *continuous* function of x and X is a *continuous* r.v., thus $P[X = x] = 0$!!!

Therefore, the distribution function $F_Y(y)$ in (4.2) becomes:

$$\begin{aligned} F_Y(y) = P\{\omega \mid Y(\omega) \leq y\} &= P\{\omega \mid X(\omega) \geq x\} \\ &= 1 - P\{\omega \mid X(\omega) \leq x\} \\ &= 1 - F_X(x)\big|_{x=T^{-1}(y)} \end{aligned}$$

(2) The density function $f_Y(y)$:

Corrsponding p.d.f. $f_Y(y)$ is then:

$$\begin{aligned} f_Y(y) = \frac{dF_Y(y)}{dy} &= \frac{d}{dy}[1 - F_X(x)] \\ &= \frac{dF_X(x)}{dx} \cdot \frac{dx}{dy} \\ &= -f_X(x) \cdot \frac{1}{\frac{dy}{dx}} \\ &= f_X(x) \cdot \frac{1}{\left|\frac{dy}{dx}\right|} \quad (\text{since } \tfrac{dy}{dx} < 0) \end{aligned}$$

where $x = T^{-1}(y)$.

3. $y = T[x]$ is a mixed continuous function of x (i.e. $\nearrow$ and $\searrow$):
   (Direct derivation of the p.d.f.)

Figure 4.9: A mixed transformation $T[\cdot]$.

Assumptions:

(i) X is a continuous r.v..

(ii) $T[\cdot]$ is NOT equal to a constant over any interval of x, i.e.

Figure 4.10: An example of "not allowable" transformation $T[\cdot]$.

Consider the following probability:

$$P\{\omega \mid y < Y(\omega) \leq y + \Delta y\} \tag{4.4}$$

It can be expressed as follows:

$$
\begin{aligned}
& P\left\{\omega \mid y < Y(\omega) \leq y + \Delta y\right\} \\
= \;& F_Y(y+\Delta y) - F_Y(y) \\
= \;& \int_{-\infty}^{y+\Delta y} f_Y(\alpha)d\alpha - \int_{-\infty}^{y} f_Y(\alpha)d\alpha \\
= \;& \int_{y}^{y+\Delta y} f_Y(\alpha)d\alpha \\
= \;& f_Y(y)\cdot\Delta y \qquad \text{(by the mean value theorem for small } \Delta y\text{)} \\
\equiv \;& P\left\{\omega \mid x_1 < X(\omega) \leq x_1 + \Delta x_1\right\} + P\left\{\omega \mid x_2 + \Delta x_2 \leq X(\omega) < x_2\right\} \\
& \quad + P\left\{\omega \mid x_3 < X(\omega) \leq x_3 + \Delta x_3\right\} \\
= \;& \int_{x_1}^{x_1+\Delta x_1} f_X(x)dx + \int_{x_2+\Delta x_2}^{x_2} f_X(x)dx + \int_{x_3}^{x_3+\Delta x_3} f_X(x)dx \\
= \;& f_X(x_1)\cdot\Delta x_1 + f_X(x_2)\cdot|\Delta x_2| + f_X(x_3)\cdot\Delta x_3
\end{aligned}
$$

Therefore,

$$f_Y(y)\cdot\Delta y = f_X(x_1)\cdot\Delta x_1 + f_X(x_2)\cdot|\Delta x_2| + f_X(x_3)\cdot\Delta x_3$$

$$\Rightarrow\; f_Y(y) = f_X(x_1)\cdot\left|\frac{\Delta x_1}{\Delta y}\right| + f_X(x_2)\cdot\left|\frac{\Delta x_2}{\Delta y}\right| + f_X(x_3)\cdot\left|\frac{\Delta x_3}{\Delta y}\right|$$

$$\overset{\Delta y \to 0}{\Longrightarrow}\; f_Y(y) = f_X(x_1)\cdot\frac{1}{\left|\frac{dy}{dx}\right|_{x=x_1}} + f_X(x_2)\cdot\frac{1}{\left|\frac{dy}{dx}\right|_{x=x_2}} + f_X(x_3)\cdot\frac{1}{\left|\frac{dy}{dx}\right|_{x=x_3}}$$

Generalizing the above concept, we have:

$$\mathbf{f_Y(y) = \sum_{i=1}^{m} f_X(x_i)\cdot\frac{1}{\left|\frac{dy}{dx}\right|_{x=x_i}}} \tag{4.5}$$

where $\mathbf{x_i = T^{-1}[y]}$ for $\mathbf{i = 1, 2, 3, \ldots, m}$

Note: (4.5) can be applicable to the cases #1 and #2 as well. (Check!!!)

Example 4.3

Square law detector:

Suppose a r.v. X w/ its p.d.f. $f_X(x)$ is applied to a square law detector to yield an output r.v. $Y \ni$:

$$Y = a \cdot X^2$$

where a is a positive constant, i.e. $a > 0$.

Figure 4.11: A square law detector $y = ax^2$.

Determine the p.d.f. $f_Y(y)$ of the output r.v. $Y(\omega)$ in terms of the input p.d.f. $f_X(x)$.

Solution:

¿From the i/o relation of the square law detector, we get the roots for a specific $y > 0$ as:

$$y = ax^2 \quad \longrightarrow \quad x = \pm\sqrt{\frac{y}{a}} \quad \longrightarrow \quad \text{let } x_1 = \sqrt{\frac{y}{a}}, \quad x_2 = -\sqrt{\frac{y}{a}}$$

Therefore,

$$\left|\frac{dy}{dx}\right| = |2ax| = 2a\,|x| \implies \begin{cases} \left|\frac{dy}{dx}\right|_{x=x_1} = 2a\sqrt{\frac{y}{a}} = 2\sqrt{ay} \\ \left|\frac{dy}{dx}\right|_{x=x_2} = 2a\sqrt{\frac{y}{a}} = 2\sqrt{ay} \end{cases}$$

Then, from (4.5), we get $f_Y(y)$ as:

(i) $y > 0$

$$\begin{aligned} f_Y(y) = &= f_X\left(\sqrt{\frac{y}{a}}\right) \cdot \frac{1}{2\sqrt{ay}} + f_X\left(-\sqrt{\frac{y}{a}}\right) \cdot \frac{1}{2\sqrt{ay}} \\ &= \frac{1}{2\sqrt{ay}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right] \end{aligned}$$

(ii) $y \leq 0$

$$f_Y(y) = 0$$

OR

¿From the definition of the distribution function, we have:

$$
\begin{aligned}
F_Y(y) = P(T \le y) &= P(x_2 leq X \le x_1) \\
&= F_X(x_1) - F_X(x_2) \\
&= F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right) \quad \text{for } y > 0
\end{aligned}
$$

Therefore, the p.d.f. $f_Y(y)$ can be derived by taking the derivative of $F_Y(y)$ as:

$$
\begin{aligned}
f_Y(y) = \frac{d}{dy}F_Y(y) &= f_X\left(\sqrt{\frac{y}{a}}\right) \cdot \frac{1}{2\sqrt{ay}} - f_X\left(-\sqrt{\frac{y}{a}}\right) \cdot \left(-\frac{1}{2\sqrt{ay}}\right) \\
&= \frac{1}{2\sqrt{ay}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right]
\end{aligned}
$$

which provides the same result!!!

Example 4.4

Fullwave rectifier:

Repeat the above example for a fullwave rectifier where the i/o relationship is as follows:

$$Y = |X|$$

Figure 4.12: A fullwave rectifier $y = |x|$.

Solution:

¿From the i/o relation of the fullwave rectifier, we get the roots for a specific $y > 0$ as:

$$y = |x| \quad \longrightarrow \quad x = \pm y \quad \longrightarrow \quad \text{let } x_1 = y, \;\; x_2 = -y$$

Therefore,

$$\left|\frac{dy}{dx}\right| = |\pm 1| \implies \begin{cases} \left|\frac{dy}{dx}\right|_{x=x_1} & = |1| \qquad = 1 \\ \left|\frac{dy}{dx}\right|_{x=x_2} & = |-1| \;\; = 1 \end{cases}$$

Then, from (4.5), we get $f_Y(y)$ as:

(i) $y > 0$

$$f_Y(y) = f_X(y)\frac{1}{1} + f_X(-y)\frac{1}{1} = f_X(y) + f_X(-y)$$

(ii) $y \le 0$

$$f_Y(y) = 0$$

Example 4.5

Saturated amplifier:

Repeat the above example for a saturated amplofier where the i/o relationship is as in the following figure ∋::

$$Y = \begin{cases} -b, & X < -a \\ \frac{b}{a}X, & -a \le X < a \\ b, & X \ge a \end{cases}$$

Figure 4.13: A saturated amplifier.

Solution:

We will first get the distribution function of Y, and then the p.d.f. by taking the derivative:

(i) $-b \leq y < b$

$$F_Y(y) = P\{\omega \mid Y(\omega) \leq y\} = P\left\{\omega \mid X(\omega) \leq \frac{a}{b}y\right\} \triangleq F_X\left(\frac{a}{b}y\right)$$

$$\longrightarrow \quad f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X\left(\frac{a}{b}y\right) = f_X\left(\frac{a}{b}y\right) \cdot \frac{a}{b}$$

(ii) $y \geq b$

$$F_Y(y) = P\{\omega \mid Y(\omega) \leq y\} = P\{\omega \mid X(\omega) \leq \infty\} = F_X(\infty) = 1$$

$$\longrightarrow \quad f_Y(y) = \frac{d}{dy}F_Y(y) = 0$$

(iii) $y < -b$

$$F_Y(y) = P\{\omega \mid Y(\omega) \leq y\} = P\{\omega \mid X(\omega) \leq -\infty\} = F_X(-\infty) = 0$$

$$\longrightarrow \quad f_Y(y) = \frac{d}{dy}F_Y(y) = 0$$

(cf.)

Notice that for the cases of (ii) and (iii) above, there is NO intersection for a specific y within that interval, and thus the p.d.f. $f_Y(y) = 0$. On the other hand, for the case of (i), we have a root $x = \frac{a}{b}y$ for a specific value y, where $\left|\frac{dy}{dx}\right| = \frac{b}{a}$, and thus from (4.5), we have:

$$f_Y(y) = \frac{f_X\left(\frac{a}{b}y\right)}{\frac{b}{a}} = f_X\left(\frac{a}{b}y\right) \cdot \frac{a}{b}$$