Contents

Chapter 7

Random Processes - Temporal Characteristics

7.1 Intorduction

There $\exists$ two kinds of signals(or processes) we deal with in engineering problems:

(1) Detrministic processes

(2) Random (Stochastic) processes

(1) Deterministic processes:

A deterministic process is a signal whose characteristics are completely known!!!

$\Longrightarrow$ It can be expressed in an exact mathematical way.

$\Longrightarrow$ The *history* and the *future behavior* (or trajectory) is transparent, i.e. completely known and can be exactly predicted.

Example 7.1

The output signal of a waveform generator: (e.g.) sine wave, saw-tooth wave etc..

$$x(t) = \sin(t)$$

Figure 7.1: Generation of a sinusoidal wave.

(cf.) $x(t_1)$ at any time t_1 is exactly known!!!

(2) Random (or Stochastic) processes:

A random process is a signal whose behavior (or value) cannot *exactly* predicted from past values!

$\Longrightarrow$ Thus, only can be described in a **probabilistic (statistical)** sense.

Example 7.2

(i) The bit stream of a binary communication system.

(e.g.)

$$P\left[x(t)|_{t=t_1} = 0\right] = p$$

$$P\left[x(t)|_{t=t_1} = 1\right] = 1 - p$$

(ii) Noises in a communication channel.

(e.g.)

$$E[n(t_1)] = m_N$$

$$\text{Var}[n(t_1)] = \sigma_N^2$$

$$P\left(n_1 < n(t) < n_2\right) \leq 1 - \alpha$$

where the *confidence level* α is given, and we want to find the corrsponding ranges n_1 and n_2 of $n(t)$.

QUESTION:

How do we represent the random processes in a systematic mathematical way?

7.2 The random process concept

Figure 7.2: The concept of random process evolved from random variable.

Characteristics:

(1) Random variable $X(\omega)$:

A random variable (r.v.) is a function of elements (ω: outcome of an experiment) in sample space S

(2) Random process $X(\omega, t)$:

A random process (r.p.) is a function of both ω and t, i.e. it represents the family or ensemble of time functions.

Notational representation:

(i) Random variable: $X(\omega) \xrightarrow{\text{abbr.}} X \xrightarrow{\text{fix } \omega} x$ (specific value of X)

(ii) Random process: $X(\omega, t) \xrightarrow{\text{abbr.}} X(t) \xrightarrow{\text{fix } \omega} x(t)$ (specific time function)[1]

(cf.) Experimental outcome:

(i) A r.v.: a value (number)

(ii) A r.p.: a function of time

[1] The specific time function $x(t)$ is called the "*sample function*".

Example 7.3

Output signals of random noise generator:

Figure 7.3: Sample functions from a random noise generator.

Special cases of r.p. $X(\omega, t)$**:**

(a) ω is fixed (i.e. specific experiment):

$X(\omega, t)$ is a specific time function: *"sample function"*

(b) time t is fixed, i.e. $t = t_1$:

$X(\omega, t)$ is a *"random variable"*

(c) both t and ω are fixed:

$X(\omega, t)$ is merely a *"number"*

Definition of Random Process:

A random process is a "family of random variables", $\{X_1, X_2, X_3, \ldots\ldots\}$.

Classification of random process:

Criteria:

1. characteristics of t: *parameter*
   (i) continuous
   (ii) discrete
2. characteristics of $X(t)$ for a fixed t (i.e. X): *random variable*
   (i) continuous
   (ii) discrete

(1) Continuous process w/ continuous parameter(t):

: Both X and t are continuous

Figure 7.4: A sample function of a continuous random process.

(cf.)

(i) It is called a "continuous random process"

(ii) It is in the form of continuous signal.

(iii) A typical example is the random noise $\ni$: communication channel noise.

(2) Continuous process w/ discrete parameter(t):

: X is continuous, but t is discrete

Figure 7.5: A sample function of a continuous random sequence.

(cf.)

(i) It is called a "continuous random sequence"

(ii) It is in the form of discrete signal.

(iii) Usually it comes from sampling the continuous random process.

(3) Discrete process w/ continuous parameter(t):

: X is discrete, but t is continuous

Figure 7.6: A sample function of a discrete random process.

(cf.)

(i) It is called a "discrete random process"

(ii) A typical example is the Poisson process.

(4) Discrete process w/ discrete parameter(t):

: Both X and t are discrete

Figure 7.7: A sample function of a discrete random sequence.

(cf.)

(i) It is called a "discrete random sequence"

(ii) It is in the form of digital signal.

(iii) Usually it comes from sampling the discrete random process.

Note:
Mostly, we deal with processes of type (1) and (3), i.e. the continuous random process and the discrete random process !!!

Example 7.4

A typical representation of a random process:

$$X(t) = A\cos(\omega t + \Theta)$$

where A, ω, and Θ could be *random variables.*

7.3 Stationarity and Independence

Idea (Bachground or intuition):

Figure 7.8: The sample functions of a random process $X(\omega, t)$.

If each one and/or combinations of random variables X_i $(i = 1, 2, 3, \ldots, M, \ldots)$ possess the **same statistical characteristics**, the random process $X(t)$ is called a **stationary** process !!!

(i) $\{X_i\}_{i=1,\ldots}$, $\{X_i, X_j\}_{i,j=1,\ldots}$ etc..

(ii) Mean, variance, joint moments etc. : statistical characteristics

$\Longrightarrow$ Depending on the degree (or order) of statistical characteristics, we categorize stationarity $\ni$: first order stationarity, second order stationarity (e.g. WSS: wide sense stationarity), upto the strict sense stationarity (SSS) with the highest order possible.

7.3.1 Prerequisites

(1) Distrubution and density functions (of a r.p. $X(t)$)

Definition 7.1 The (fist order) distribution function of a random process $X(t)$ at time $t = t_1$ (i.e. random variable X_1) is defined as: [2]

$$F_X(x_1; t_1) \triangleq P\left[X(t_1) \leq x_1\right] \ : \ \text{1st order distribution}$$

where x_1 is a real number.

Definition 7.2 Similarly, the N-th order joint distribution function of a random process $X(t)$ at times $t_1, t_2, \ldots, t_N$ is defined as:

$$\begin{aligned} F_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) \ &\triangleq \ P\left[\{X(t_1) \leq x_1\} \cap \cdots \cap \{X(t_N) \leq x_N\}\right] \\ &\quad : \text{N-th order distribution} \end{aligned}$$

where $x_1, x_2, \ldots, x_N$ are real numbers.

Definition 7.3 Corresponding probability density functions are defined as derivatives of the distribution functions:

$$f_X(x_1; t_1) \triangleq \frac{dF_X(x_1; t_1)}{dx_1}$$

$$\vdots$$

$$f_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) \triangleq \frac{\partial^N F_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N)}{\partial x_1 \partial x_2 \cdots \partial x_N}$$

[2] Note that $X(t_1) = X_1$ is a random variable, and the definition of the 1st order distribution function of a r.p. comes from the definition of the probability distribution function of a r.v..

(2) Statistical independence (of random processes)

Definition 7.4 Two random processes $X(t)$ and $Y(t)$ are called statistically independent if random vectors $\{X(t_1), X(t_2), \cdots, X(t_N)\}$ and $\left\{Y(t_1'), Y(t_2'), \cdots, Y(t_N')\right\}$ are independent, i.e. **if:**

$$\begin{aligned} & f_{XY}\left(x_1, \ldots, x_N, y_1, \ldots, y_M; t_1, \ldots, t_N, t_1', \ldots, t_M'\right) \\ = \; & f_X\left(x_1, \ldots, x_N; t_1, \ldots, t_N\right) \cdot f_Y\left(y_1, \ldots, y_M; t_1', \ldots, t_M'\right) \end{aligned}$$

7.3.2 First order stationary random process

Definition 7.5 A random process $X(t)$ is called to be *1st order stationary* if for any t_1 and Δ;

$$f_X(x; t_1) = f_X(x; t_1 + \Delta)$$

i.e. the probability density function (p.d.f.) is invariant under time shift!!!

FACT:

If a r.p. $X(t)$ is 1st order stationay, the *mean* is constant, i.e. independent of time: [3]

$$E[X(t)] = \text{constant} \triangleq \overline{X}$$

proof:

Choose any two arbitrary times t_1 and t_2 along the r.p. $X(t)$, and let:

$$t_2 = t_1 + \Delta$$

Then, we have:

$$\begin{aligned} E[X(t_2)] &= E[X(t_1 + \Delta)] \\ &= \int_{-\infty}^{\infty} x f_X(x; t_1 + \Delta) dx \\ &= \int_{-\infty}^{\infty} x f_X(x; t_1) dx \\ &= E[X(t_1)] \end{aligned}$$

i.e. we have

$$E[X(t_1 + \Delta)] = E[X(t_1)] = \text{constant}$$

since t_1 and Δ are assumed to be arbitrary.

(cf.) In the above proof, we have used the following definition of the expectation of a r.p. at time t_1, which is the expectation of a random variable $X(t_1) = X_1$:

$$E[X(t_1)] = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

[3] Note that the reverse does not hold, i.e. if the *mean* of a r.p. is constant, that does not necessarily mean that the r.p. is 1st order stationary.

7.3.3 Second order and wide sense stationarity

Definition 7.6 A random process $X(t)$ is called to be *second order stationary* if for any t_1, t_2 and Δ;

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

Figure 7.9: A sample function of the 2nd order stationary r.p..

NOTE:

The joint distribution function of $X(t)$ at two time points t_1 and t_2 depends *only* on the time difference $\tau \triangleq t_2 - t_1$, i.e.

$$\begin{aligned} f_X(x_1, x_2; t_1, t_2) &= f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) \\ &\xrightarrow{\Delta = -t_1} f_X(x_1, x_2; 0, \tau) \\ &= f_X(x_1, x_2; \tau) \end{aligned}$$

Definition 7.7 The autocorrelation function of a random process $X(t)$ at time t_1 and t_2 is defined as follows; [4]

$$R_{XX}(t_1, t_2) \triangleq E\left[X(t_1)X(t_2)\right]$$

(cf.) Note that this is the correlation of two random variables $X(t_1)$ and $X(t_2)$.

Fact:

The autocorrelation function of a second order stationary random process $X(t)$ is a function of only $\tau = t_2 - t_1$!!!

proof:

$$\begin{aligned}
R_{XX}(t_1, t_2) \triangleq E\left[X(t_1)X(t_2)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; \tau)dx_1 dx_2 \\
&= R_{XX}(\tau)
\end{aligned}$$

: function of τ only

More relaxed form of the second order stationarity:
$\longrightarrow$ **wide sense stationarity (WSS)**

Definition 7.8 A random process $X(t)$ is called a WSS process **if:**

(i) $E\left[X(t)\right] = \text{constant}$

(ii) $E\left[X(t_1)X(t_2)\right] = R_{XX}(\tau)$ where $\tau = t_2 - t_1$.

Remark:
Notice that the conditions on WSS are only in terms of the expected values, NOT on the distribution or density functions of $X(t)$!!!

---

[4] This will form the base concept for the definition of the WSS(wide sense stationary) random process!

Note: The relationship among 1st order, 2nd order, and wide sense stationarities:

Figure 7.10: Relationship among stationarities.

Example 7.5

Determine whether the following r.p. $X(t)$ is WSS or not, for each given case:

$$X(t) = A\cos(\omega_0 t + \Theta)$$

(1) $A \sim U[0,1]$ and ω_0 & Θ are contants.
(2) $\omega_0 \sim U[0,W]$ and A & Θ are contants.
(3) $\Theta \sim U[0,2\pi]$ and A & ω_0 are contants.

Solution:

(1) $A \sim U[0,1]$ and ω_0 & Θ are contants.

(i) Mean:

$$\begin{aligned} E[X(t)] &= \int_0^1 a\cos(\omega_0 t + \theta) f_A(a)da \\ &= \left[\frac{a^2}{2}\right]_0^1 \cos(\omega_0 t + \theta) \\ &= \frac{1}{2}\cos(\omega_0 t + \theta) \\ &\quad : \text{ depends on } t \end{aligned}$$

(ii) Autocorrelation:

$$
\begin{aligned}
R_{XX}(t_1, t_2) &= \int_0^1 a^2 \cos(\omega_0 t_1 + \theta) \cos(\omega_0 t_2 + \theta) f_A(a) da \\
&= \frac{1}{3} \cos(\omega_0 t_1 + \theta) \cos(\omega_0 t_2 + \theta) \\
&= \frac{1}{6} \left\{ \cos\left[\omega_0 (t_1 + t_2) + 2\theta\right] + \cos\left[\omega_0 (t_1 - t_2)\right] \right\} \\
&\quad : \text{ depends on } t_1 \text{ and } t_2
\end{aligned}
$$

$\Longrightarrow$ $X(t)$ is NOT WSS!

(2) $\omega_0 \sim U[0, W]$ and A & Θ are contants.

(i) Mean:

$$
\begin{aligned}
E[X(t)] &= \frac{1}{W} \int_0^W A \cos(\omega_0 t + \theta) d\omega_0 \\
&= \frac{A}{W} \left[\frac{\sin(\omega_0 t + \theta)}{t} \right]_0^W \\
&= \frac{A}{Wt} \left\{ sin(Wt + \theta) - sin(\theta) \right\} \\
&\quad : \text{ depends on } t
\end{aligned}
$$

(ii) Autocorrelation:

$$
\begin{aligned}
R_{XX}(t_1, t_2) &= \frac{1}{W} \int_0^W A^2 \cos(\omega_0 t_1 + \theta) \cos(\omega_0 t_2 + \theta) d\omega_0 \\
&= \frac{A^2}{2W} \int_0^W \left\{ \cos\left[\omega_0 (t_1 + t_2) + 2\theta\right] + \cos\left[\omega_0 (t_1 - t_2)\right] \right\} d\omega_0 \\
&= \frac{A^2}{2W} \left. \frac{\sin[\omega_0 (t_1 + t_2) + 2\theta]}{t_1 + t_2} \right|_0^W + \left. \frac{\sin[\omega_0 (t_1 - t_2)]}{t_1 - t_2} \right|_0^W \\
&= \frac{A^2}{2W} \left\{ \frac{\sin[W(t_1 + t_2) + 2\theta] - \sin(2\theta)}{t_1 + t_2} + \frac{\sin[W(t_1 - t_2)]}{t_1 - t_2} \right\} \\
&\quad : \text{ depends on } t_1 \text{ and } t_2
\end{aligned}
$$

$\Longrightarrow$ $X(t)$ is NOT WSS!

(3) $\Theta \sim U[0, 2\pi]$ and A & ω_0 are contants.

(i) Mean:

$$\begin{aligned} E[X(t)] &= \int_0^{2\pi} A\cos(\omega_0 t + \theta)\frac{1}{2\pi}d\theta \\ &= \frac{A}{2\pi}\sin(\omega_0 t + \theta)|_0^{2\pi} \\ &= 0 \end{aligned}$$

: independent of t

(ii) Autocorrelation:

$$\begin{aligned} R_{XX}(t_1, t_2) &= \int_0^{2\pi} A^2 \cos(\omega_0 t_1 + \theta)\cos(\omega_0 t_2 + \theta)\frac{1}{2\pi}d\theta \\ &= \frac{A^2}{2\pi}\cdot\frac{1}{2}\int_0^{2\pi}\{\cos[\omega_0(t_1 + t_2) + 2\theta] + \cos[\omega_0(t_1 - t_2)]\}\,d\theta \\ &= \frac{A^2}{4\pi}\cdot 2\pi\cos[\omega_0(t_1 - t_2)] \\ &= \frac{A^2}{2}\cos(\omega_0\tau) \end{aligned}$$

: depends only on $\tau \triangleq t_1 - t_2$

$\Longrightarrow$ $X(t)$ is wide sense stationary (WSS)!

Definition 7.9 Two random processes $X(t)$ and $Y(t)$ are called *jointly WSS* (JWSS) **if:** [5]

(i) $X(t)$ and $Y(t)$ are WSS individually.

(ii) $R_{XY}(t_1, t_2) \triangleq E[X(t_1)Y(t_2)] = R_{XY}(\tau)$
i.e. function of the time difference τ only, where $\tau = t_2 - t_1$.

[5] $R_{XY}(t_1, t_2)$ in this definition is the cross-correlation between $X(t)$ and $Y(t)$, which will be defined at later section along with its properties.

7.3.4 N-th order & strict sense stationarity : Generalization

Definition 7.10 A random process $X(t)$ is called *N-th order stationary* if its N-th order probability density function is independent of the absolute time, i.e.

$$f_X(x_1, x_2, \ldots, x_N; t_1, t_2, \ldots, t_N) = f_X(x_1, x_2, \ldots, x_N; t_1 + \Delta, t_2 + \Delta, \ldots, t_N + \Delta)$$

$$\forall\ t_i \text{ and } \Delta \quad i = 1, 2, \ldots, N$$

Note:

N-th order stationarity $\xrightarrow{O}$ k-th order stationarity $\quad \forall\ k \leq N$

Definition 7.11 A random process $X(t)$ is called strict sense stationary (SSS) if it is stationary for all orders, $N = 1, 2, \ldots\ldots\ldots.$

7.3.5 Time averages and ergodicity

Definition 7.12 The time average of a function $f(t)$ is denoted and defined as follows:

$$A\left[f(t)\right] \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt$$

Note:
The notation of operator A comes as the counterpart of the mathematical expectation E:

(i) $A[\cdot]$: Time average

(ii) $E[\cdot]$: Statistical (or ensemble) average

Definition 7.13 The *mean* and the *autocorrelation function* of a random process $X(t)$, as time averages are defined as follows;

(1) Time mean:

$$\overline{x} \triangleq A\left[X(t)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt$$

(2) Time autocorrelation function:

$$\mathcal{R}_{XX}(\tau) \triangleq A\left[X(t)X(t+\tau)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt$$

(cf.) Notice that $\overline{x}$ and $\mathcal{R}_{XX}(\tau)$ varies depending on the sample function $x(t)$ of the r.p. $X(t)$.

FACT:
$\overline{x}$ and $\mathcal{R}_{XX}(\tau)$ for a fixed τ are:

(i) *constants* for a specific sample function $x(t)$. [6]

(ii) *random variables* for the random process $X(t)$. [7]

By taking expectations of the time mean and the time autocorrelation function, we have for a stationary (or at least WSS: 2nd order) random process $X(t)$:

$$E\left[\mathcal{R}_{XX}(\tau)\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[X(t)X(t+\tau)\right] dt = \lim_{T\to\infty} \frac{2T}{2T} \cdot \overline{X} = \overline{X}$$

$$E\left[\overline{x}\right] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[X(t)\right] dt = \lim_{T\to\infty} \frac{2T}{2T} \cdot R_{XX}(\tau) = R_{XX}(\tau)$$

: from which we can conclude that for a *stationary* random process $X(t)$:

E[**time average**] $=$ **statistical average**

Ergodic Theorem:

If random variables $\overline{x}$ and $\mathcal{R}_{XX}(\tau)$ have zero variances (i.e. they are constants) [8], we have:

$$E\left[\overline{x}\right] = \overline{x} \equiv \overline{X} \tag{7.1}$$

$$E\left[\mathcal{R}_{XX}(\tau)\right] = \mathcal{R}_{XX}(\tau) \equiv R_{XX}(\tau) \tag{7.2}$$

$\Longrightarrow$ *Time averages* and *statistical averages* of a r.p. $X(t)$ become equal.

$\Longrightarrow$ Then, $X(t)$ is called an *ergodic process* !!!

[6] In the same token, $\overline{x}$ and $\mathcal{R}_{XX}(\tau)$ for a fixed τ are *constants* for deterministic signals.
[7] Be reminded that $X(t)$ implies many possible sample function $x(t)$'s.
[8] For $X(t)$ to be an ergodic r.p., the time average of every sample function should be the same, i.e. independent of ω in the sample space S.

Why ergodicity?

In real world, we cannot deal with entire ensemble of $X(t)$, i.e. we only deal with one or a few sample functions of it !

$\longrightarrow$ we cannot compute statistical (i.e. ensemble) averages of $X(t)$.

$\longrightarrow$ we have to replace it by the time averages of $x(t)$.

$\longrightarrow$ we need the concept of *ergodicity* !!!

Figure 7.11: Concept of ergodocity.

Note: [9]

(i) If only (7.1) is satisfied : Mean ergodic(1st order)

(ii) If both (7.1) and (7.2) are satisfied : Variance ergodic(2nd order)

Fact:

ergodic process $\longrightarrow$ **stationary process**

[9] Most of the cases, we deal w/ the variance ergodic (i.e. 2nd order) processes.

(e.g.)

If a r.p. $X(t)$ is ergodic, then

(i) $E[X(t)] = A[X(t)] \equiv$ constant (zero variance r.v.):

$$E[X(t)] = \overline{X} = \text{constatnt} \ \ \forall\, t$$

(ii) $E[X(t)X(t+\tau)] = A[X(t)X(t+\tau)] \equiv \mathcal{R}_{XX}(\tau)$:

$$R_{XX}(t, t+\tau) = \mathcal{R}_{XX}(\tau) \ : \ \text{function of } \tau \text{ only}$$

Therefore, from (i) and (ii), $X(t)$ must be *stationary*.

Definition 7.14 Two random processes $X(t)$ and $Y(t)$ are called *jointly ergodic* **if:**

(i) $X(t)$ and $Y(t)$ are ergodic individually

(ii) Time cross-correlation is equal to the statistical cross-correlation, i.e.

$$\mathcal{R}_{XY}(\tau) \equiv R_{XY}(\tau)$$

where

$$\mathcal{R}_{XY}(\tau) \triangleq A\left[X(t)Y(t+\tau)\right] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)Y(t+\tau)dt$$

$$R_{XY}(\tau) \triangleq E\left[X(t)Y(t+\tau)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t)Y(t+\tau)f_{XY}(x,y;t,t+\tau)dxdy$$

Example 7.6

Given two random processes $X(t)$ and $Y(t)$ as follows:

$$X(t) = A\cos(\omega_0 t + \Theta)$$

$$Y(t) = B\sin(\omega_0 t + \Theta)$$

where A, B and ω_0 are constant whereas Θ is uniformly distributed random variable between 1 and 2π, i.e. $\Theta \sim [0, 2\pi]$.

Determine whether $X(t)$ and $Y(t)$ are jointly ergodic or not.

Solution:

We must prove:

(a) $X(t)$ is ergodic.

(b) $Y(t)$ is ergodic.

(c) $\mathcal{R}_{XY}(\tau) = R_{XY}(\tau)$.

(a) Ergodicity of $X(t)$:

(1) Mean:

(i) Statistical mean:

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta) \cdot \frac{1}{2\pi} d\theta = 0$$

(ii) Time mean:

$$\begin{aligned} A[X(t)] &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A\cos(\omega_0 t + \theta) dt \\ &= \lim_{T\to\infty} \frac{A}{2T} \left[\frac{\sin(\omega_0 t + \theta)}{\omega_0} \right]_{-T}^{T} \\ &= \lim_{T\to\infty} \frac{A}{2T\omega_0} \left[\sin(\omega_0 T + \theta) - \sin(-\omega_0 T + \theta) \right] \\ &= 0 \end{aligned}$$

$$\therefore \; E[X(t) = A[X(t)]$$

(2) Autocorrelation:

(i) Statistical autocorrelation:

$$\begin{aligned} R_{XX}(\tau) &= E\left[X(t)X(t+\tau)\right] \\ &\vdots \qquad \text{(derived before)} \\ &= \frac{A^2}{2}\cos(\omega_0\tau) \end{aligned}$$

(ii) Time autocorrelation:

$$\begin{aligned} \mathcal{R}_{XX}(\tau) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt \\ &= \lim_{T\to\infty}\frac{A^2}{2T}\int_{-T}^{T}\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)dt \\ &= \lim_{T\to\infty}\frac{A^2}{4T}\int_{-T}^{T}\left\{\cos(2\omega_0 t+\omega_0\tau+2\theta)+\cos(\omega_0\tau)\right\}dt \\ &= \lim_{T\to\infty}\frac{A^2}{4T}\left\{constant+2T\cos(\omega_0\tau)\right\} \\ &= \frac{A^2}{2}\cos(\omega_0\tau) \end{aligned}$$

$$\therefore\ R_{XX}(\tau) = \mathcal{R}_{XX}(\tau)$$

Therefore, $X(t)$ is an ergodic random process.

(b) Ergodicity of $Y(t)$:

Similary, we can show that $Y(t)$ is also an ergodic random process, i.e.:

(1) Mean: $E[Y(t)] = A[Y(t)]$.

(2) Autocorrelation: $R_{YY}(\tau) = \mathcal{R}_{YY}(\tau)$.

(c) Cross-correlation between $X(t)$ and $Y(t)$:

(1) Statistical cross-correlation:

$$
\begin{aligned}
R_{XY}(\tau) &= E\left[X(t)Y(t_\tau)\right] \\
&= \int_0^{2\pi} AB\cos(\omega_0 t + \theta)\sin(\omega_0 t + \omega_0\tau + \theta)\cdot\frac{1}{2\pi}d\theta \\
&= \frac{AB}{4\pi}\int_0^{2\pi}\left\{\sin(2\omega_0 t + \omega_0\tau + 2\tau) + \sin(\omega_0\tau)\right\}d\theta \\
&= \frac{AB}{4\pi}\cdot 2\pi\sin(\omega_0\tau) \\
&= \frac{AB}{2}\sin(\omega_0\tau)
\end{aligned}
$$

(2) Time cross-correlation:

$$
\begin{aligned}
\mathcal{R}_{XY}(\tau) &= \lim_{T\to\infty}\frac{AB}{2T}\int_{-T}^{T}\cos(\omega_0 t + \theta)\sin(\omega_0 t + \omega_0\tau + \theta)dt \\
&= \lim_{T\to\infty}\frac{AB}{4T}\int_{-T}^{T}\left\{\sin(2\omega_0 t + \omega_0\tau + 2\theta) + \sin(\omega_0\tau)\right\}dt \\
&= \lim_{T\to\infty}\frac{AB}{4T}\left\{constant + 2T\sin(\omega_0\tau)\right\} \\
&= \frac{AB}{2}\sin(\omega_0\tau)
\end{aligned}
$$

$$\therefore\ R_{XY}(\tau) = \mathcal{R}_{XY}(\tau)$$

Therefore, we can conclude that $X(t)$ and $Y(t)$ are **jointly ergodic** random processes !!!

7.4 Correlation functions

7.4.1 Autocorrelation function & properties

Recall that the autocorrelation function $R_{XX}(t_1, t_2)$ of a r.p. $X(t)$ at times t_1 and t_2 has been defined as:

$$R_{XX}(t_1, t_2) \triangleq E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1 dx_2$$

Figure 7.12: Concept of the autocorrelation of a r.p. $X(t)$.

Suppose $X(t)$ is a WSS random process, and let $t_1 = t$, $t_2 = t + \tau$, then:

$$\begin{aligned} R_{XX}(t_1, t_2) &= R_{XX}(t, t+\tau) \\ &= E\left[X(t)X(t+\tau)\right] \\ &= R_{XX}(\tau) \end{aligned}$$

i.e. the autocorrelation function of a WSS r.p. $X(t)$ at two time instants t_1, t_2 depends only on the time difference $t_2 - t_1 \triangleq \tau$: *function of τ only!!!.*

Properties: (of $R_{XX}(\tau)$ for a WSS r.p. $X(t)$) [10]

(1) $|R_{XX}(\tau)| \leq R_{XX}(0)$ (i.e. $R_{XX}(0)$ is the maximum.)

(2) $R_{XX}(-\tau) = R_{XX}(\tau)$ (i.e. $R_{XX}(\tau)$ is symmetric.)

(3) $R_{XX}(0) = E[X^2(t)] \geq 0$ (i.e. $R_{XX}(0)$ is the power of $X(t)$.)

proof:

(1) $|R_{XX}(\tau)| \leq R_{XX}(0)$

Let $Y(t) = X(t) \pm X(t+\tau)$, then we have:

$$\begin{aligned} E[Y^2(t)] &= E\left[X^2(t) \pm 2X(t)X(t+\tau) + X^2(t+\tau)\right] \\ &= R_{XX}(0) \pm 2R_{XX}(\tau) + R_{XX}(0) \\ &\geq 0 \quad \text{(should be !)} \end{aligned}$$

Therefore,

$$-R_{XX}(0) \leq R_{XX}(\tau) \leq R_{XX}(0)$$

$$\Longrightarrow \quad |R_{XX}(\tau)| \leq R_{XX}(0)$$

(2) assignment

(3) assignment

[10] Notice that these properties are same as those for the deterministic signals discussed in Signals and Systems class!

Other properties:

(4) If $\overline{X} \neq 0$ and $X(t)$ is not periodic, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$. [11]

(5) If $X(t)$ is periodic (T), then $R_{XX}(\tau)$ is also periodic (T).

(6) If $X(t)$ is zero mean, ergodic r.p., and has no periodic components, then $\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$.

Figure 7.13: A periodic r.p. $X(t)$ such as $X(t) = A\cos(\omega_0 t + \Theta)$.

Example 7.7

Assume that a WSS r.p. $X(t)$ has the autocorrelation function as follows:

$$R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2}$$

Then, find the mean and the variance of $X(t)$.

Solution:

(i) Mean:
¿From the property (4), we have:

$$\lim_{\tau\to\infty} R_{XX}(\tau) = \overline{X}^2 = 25 + \lim_{\tau\to\infty} \frac{4}{1 + 6\tau^2} = 25$$

Therefore, the mean is:

$$\therefore \ \overline{X} = 5$$

[11] As $\tau \to \infty$, $X(t)$ and $X(t+\tau)$ become independent (or uncorrelated), and thus $R_{XX}(\tau) = E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)]$. If $X(t)$ is periodic, they cannot be independent (or uncorrelated).

(ii) Variance:

$$\begin{aligned}
\sigma_X^2(t) &\triangleq E\left[X^2(t)\right] - \overline{X}^2 \\
&= R_{XX}(0) - 25 \\
&= 29 - 25 \\
&= 4 \\
&= \sigma_X^2 \quad : \text{independent of time}
\end{aligned}$$

7.4.2 Cross-correlation function & properties

Recall that the cross-correlation function $R_{XY}(t_1, t_2)$ of r.p. $X(t)$ and $Y(t)$ at times t_1 and t_2 has been defined as:

$$R_{XY}(t_1, t_2) \triangleq E\left[X(t_1)Y(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y; t_1, t_2)dxdy$$

Let $t_1 - t$ and $t_2 = t_1 + \tau$, then we have:

$$R_{XY}(t_1, t_2) = R_{XY}(t, t+\tau) \triangleq E\left[X(t)Y(t+\tau)\right]$$

(1) Jointly WSS: [12]

$$R_{XY}(t, t+\tau) = R_{XY}(\tau) \quad : \text{function of } \tau \text{ only}$$

(2) Orthogonal:

$$R_{XY}(t, t+\tau) = 0 \quad \forall\, t \text{ and } \tau$$

[12] Also $X(t)$ and $Y(t)$ should be WSS individually.

(3) Statistical independence:

If $x(t)$ and $Y(t)$ are statistically independent, then

$$R_{XY}(t, t+\tau) = E\left[X(t)Y(t+\tau)\right] = E\left[X(t)\right]E\left[Y(t+\tau)\right]$$

(cf.) Combining (1) and (3), i.e. if $X(t)$ and $Y(t)$ are jointly WSS and statistically independent,

$$R_{XY}(\tau) = \overline{X} \cdot \overline{Y} = \text{constant}$$

Properties: (of $R_{XY}(\tau)$ for WSS r.p. $X(t)$ and $Y(t)$)

(1) $|R_{XY}(-\tau)| = R_{XY}(\tau)$ (i.e. $R_{XY}(\tau)$ is anti-symmetric.)

(2) $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$ (i.e. bounded by geometric mean.)

(3) $|R_{XY}(\tau)| \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$ (i.e. bounded by algebraic mean.)

proof: assignment

Note:

Notice that the geometric mean of $R_{XX}(0)$ and $_{YY}(0)$ in (2) provides tighter upper bound of $R_{XY}(\tau)$ than the algebraic mean in (3), since:

$$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}\left[R_{XX}(0) + R_{YY}(0)\right]$$

Example 7.8

Given two r.p.'s $X(t)$ and $Y(t)$ as follows:

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

where ω_0 is a constant, and A, B are *uncorrelated* zero mean random variables with the same variance σ^2.

Determine whether $X(t)$ and $Y(t)$ are JWSS or not.

Solution:

We must prove:

(a) $X(t)$ and $Y(t)$ are WSS individually.
(b) $R_{XY}(t, t+\tau) = R_{XY}(\tau)$: function of τ only !

¿From the given conditions, we have the followng facts:

(i) Since A, B are uncorrelated and have zero means:

$$E[AB] = E[A] \cdot E[B] = 0$$

.

(ii) Since they have the same variance, and zero means:

$$E[A^2] = E[B^2] = \sigma^2$$

.

Also, recall the following triginometric relationships:

(iii) $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.
(iv) $\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta)\cos(\alpha)\sin(\beta)$.

(1) $X(t)$ is WSS:

(i) Mean:

Since $E[A] = E[B] = 0$, we have

$$\begin{aligned} E\left[X(t)\right] &= E\left[A\cos(\omega_0 t) + B\sin(\omega_0 t)\right] \\ &= E[A]\cos(\omega_0 t) + E[B]\sin(\omega_0 t) \\ &= 0 \quad : \text{constant} \end{aligned}$$

(ii) Autocorrelation function:

$$\begin{aligned} & R_{XX}(t, t+\tau) \\ = \;& E\left[X(t)X(t+\tau)\right] \\ = \;& E\left[\{A\cos(\omega_0 t) + B\sin(\omega_0 t)\}\{A\cos(\omega_0 t + \omega_0\tau) + B\sin(\omega_0 t + \omega_0\tau)\}\right] \\ = \;& E[A^2]\cos(\omega_0 t)\cos(\omega_0 t + \omega_0\tau) \\ & +E[AB]\{\cos(\omega_0 t)\sin(\omega_0 t + \omega_0\tau) + \sin(\omega_0 t)\cos(\omega_0 t + \omega_0\tau)\} \\ & +E[B^2]\sin(\omega_0 t)\sin(\omega_0 t + \omega_0\tau) \\ = \;& \sigma^2\cos(\omega_0 t + \omega_0\tau - \omega_0 t) \\ = \;& \sigma^2\cos(\omega_0\tau) \quad : \text{function of } \tau \text{ only} \end{aligned}$$

Therefore, $X(t)$ is WSS.

(2) $Y(t)$ is WSS:

(i) Mean:

$$\begin{aligned} E\left[Y(t)\right] &= E\left[B\cos(\omega_0 t) - A\sin(\omega_0 t)\right] \\ &= E[B]\cos(\omega_0 t) - E[A]\sin(\omega_0 t) \\ &= 0 \quad : \text{constant} \end{aligned}$$

(ii) Autocorrelation function:

$$
\begin{aligned}
& R_{YY}(t, t+\tau) \\
= \; & E\left[Y(t)Y(t+\tau)\right] \\
= \; & E\left[\{B\cos(\omega_0 t) - A\sin(\omega_0 t)\}\{B\cos(\omega_0 t + \omega_0 \tau) - A\sin(\omega_0 t + \omega_0 \tau)\}\right] \\
= \; & E[B^2]\cos(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) + E[A^2]\sin(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau) \\
= \; & \sigma^2 \cos(\omega_0 t + \omega_0 \tau - \omega_0 t) \\
= \; & \sigma^2 \cos(\omega_0 \tau) \quad : \text{function of } \tau \text{ only}
\end{aligned}
$$

Therefore, $Y(t)$ is WSS.

(3) Cross-correlation between $X(t)$ and $Y(t)$:

$$
\begin{aligned}
& R_{XY}(t, t+\tau) \\
= \; & E\left[X(t)Y(t+\tau)\right] \\
= \; & E\left[\{A\cos(\omega_0 t) + B\sin(\omega_0 t)\}\{B\cos(\omega_0 t + \omega_0 \tau) - A\sin(\omega_0 t + \omega_0 \tau)\}\right] \\
= \; & E[B^2]\sin(\omega_0 t)\cos(\omega_0 t + \omega_0 \tau) - E[A^2]\cos(\omega_0 t)\sin(\omega_0 t + \omega_0 \tau) \\
= \; & \sigma^2 \cos(\omega_0 t - \omega_0 t - \omega_0 \tau) \\
= \; & -\sigma^2 \sin(\omega_0 \tau) \quad : \text{function of } \tau \text{ only}
\end{aligned}
$$

Therefore, $X(t)$ and $Y(t)$ are jointly WSS (JWSS) !!!

7.4.3 Covariance functions

Definition 7.15 The auto-covariance function of a r.p. $X(t)$ is defined as:

$$C_{XX}(t, t+\tau) \triangleq E\left[\left(X(t) - E[X(t)]\right)\left(X(t+\tau) - E[X(t+\tau)]\right)\right]$$

another form:

$$\begin{aligned} C_{XX}(t, t+\tau) &= E\left[X(t)X(t+\tau)\right] - E\left[X(t)\right]E\left[X(t+\tau)\right] \\ &= R_{XX}(t, t+\tau) - E\left[X(t)\right]E\left[X(t+\tau)\right] \end{aligned} \tag{7.3}$$

Definition 7.16 The cross-covariance function of r.p.'s $X(t)$ and $Y(t)$ is defined as:

$$C_{XY}(t, t+\tau) \triangleq E\left[\left(X(t) - E[X(t)]\right)\left(Y(t+\tau) - E[Y(t+\tau)]\right)\right]$$

or

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - E\left[X(t)\right]E\left[Y(t+\tau)\right] \tag{7.4}$$

Note:
If $X(t)$ and $Y(t)$ are JWSS, then (7.3) and (7.4) become:

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

and

$$X_{XY}(\tau) = R_{XY}(\tau) - \overline{X} \cdot \overline{Y}$$

Note:
The variance of a WSS r.p. $X(t)$ is the value of $C_{XX}(\tau)$ at $\tau = 0$, i.e.:

$$\begin{aligned} \sigma_X^2 &\triangleq E\left[X^2(t)\right] - E\left[X(t)\right]^2 \\ &= R_{XX}(0) - \overline{X}^2 \\ &\triangleq C_{XX}(0) \end{aligned}$$

Definition 7.17 Two random processes $X(t)$ and $Y(t)$ are called *uncorrelated* **if:** [13]

$$C_{XY}(t, t+\tau) = 0$$

or, equivalently

$$R_{XY}(t, t+\tau) = E\left[X(t)\right] \cdot E\left[X(t+\tau)\right]$$

Remark:

For two random processes $X(t)$ and $Y(t)$,

$$\text{statistically independent} \quad \underset{\underset{X}{\longleftarrow}}{\overset{\text{O}}{\longrightarrow}} \quad \text{uncorrelated}$$

(cf.) The reverse is ONLY valid when $X(t)$ and $Y(t)$ are jointly Gaussian random processes ! [14]

[13]Caution: *"uncorrelatedness"* means $C_{XY}(t, t+\tau) = 0$, NOT $R_{XY}(t, t+\tau) = 0$.
[14]To be discussed later.

7.5 Measurement of correlation functions

In real world, we cannot measure correlations in statistical sense, since we cannot have all of the ensemble of $X(t)$.

$\Longrightarrow$ We have to resort to time averages of a specific sample function $x(t)$.

$\Longrightarrow$ The process $X(t)$ must be assumed to *ergodic* like it or not.

$\Longrightarrow$ Moreover, the observation time must be limited!

: approximation needed !!!

Figure 7.14: A specific sample function $x(t)$ with limited observation time.

Block diagram:

Figure 7.15: A block diagram of measuring correlation of random processes.

Assumption: $X(t)$ and $Y(t)$ are *jointly ergodic*!!! [15]

Analysis:

The output of the system at time $t = t_1 + 2T$, where t_1 is arbitrary, is:

$$R_0(t_1 + 2T) = \frac{1}{2T}\int_{t_1}^{t_1+2T} x(t-T)y(t-T+\tau)dt \tag{7.5}$$

Let $t^{'} = t - T$, then:

$$R_0(t_1 + 2T) = \frac{1}{2T}\int_{t_1-T}^{t_1+T} x(t^{'})y(t^{'}+\tau)dt^{'}$$

Choose $t_1 = 0$, [16] then: [17]

$$\begin{aligned} R_0(2T) &= \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt \\ &\approx \mathcal{R}_{XY}(\tau) \quad : \text{time correlation function (approximation)} \\ &= R_{XY}(\tau) \quad (\text{since } X(t) \text{ and } Y(t) \text{ are jointly ergodic}) \end{aligned}$$

$\Longrightarrow$ *" Repeat with different τ until all of the desired range of τ is covered ! "* (e.g. $0 \le \tau \le T$) [18]

[15]This assumption is for replacing statistical averages of r.p. $X(t)$ with time averages of a sample function $x(t)$.

[16]Since jointly ergodic means the JWSS, and $\int_{t_1-T}^{t_1+T}$ should be independent of t_1 for large T.

[17]Recall $\mathcal{R}_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)y(t+\tau)dt$.

[18]Refer (7.5).

7.6 Gaussian random processes

Among various randomprocesses, one of the most important aand frequently used r.p. is the Gaussian random process.

Definition 7.18 A random process $X(t)$ is called *Gaussian* if for any $N = 1, 2, \ldots$ and given times $t_1, t_2, \ldots, t_N$, the random vector $\underline{X} \triangleq (X_1, X_2, \ldots, X_N)^T$ are jointly Gaussian, where $X_i = X(t_1)$,
i.e., the joint probability density function must be in the following form: [19] [20]

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = \frac{1}{\sqrt{(2\pi)^N |\mathbf{C}_X|}} \exp\left\{-\frac{1}{2}(\underline{x} - \overline{\underline{X}})^T \mathbf{C}_X^{-1} (\underline{x} - \overline{\underline{X}})\right\}$$

where

(1) $\underline{x}$ is a specific vector of $\underline{X}$:

$$\underline{x} = (x_1, x_2, \ldots, x_N)^T$$

(2) $\overline{\underline{X}}$ is the mean vector:

$$\overline{\underline{X}} = \left(\overline{X_1}, \overline{X_2}, \ldots, \overline{X_N}\right)^T$$

(3) $\mathbf{C}_X$ is the $N \times N$ covariance matrix:

$$\mathbf{C}_X = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

where

$$\begin{aligned} C_{ik} \triangleq C_{X_i X_k} &= E\left[(X_i - \overline{X_i})(X_k - \overline{X_k})\right] \\ &= C_{XX}(t_i, t_k) \\ &\quad : \text{autocovariance of } X(t) \text{ at } t = t_1 \text{ and } t = t_k \end{aligned}$$

[19] Notice that the only two quantities we need to completely define a Gaussian r.p. are the mean vector $\overline{\underline{X}}$ and the covariance matrix $\mathbf{C}_X$.

[20] Recall that the p.d.f. of a Gaussian r.v. is:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\overline{x})^2}{2\sigma^2}}$$

NOTE:
We only need the *mean* and *autocovariance functions* (or autocorrelation) of $X(t)$ to *completely* specify a Gaussian random process !!! [21]

Remark:

If $X(t)$ is a WSS Gaussian random process, then: [22]

(i) $\overline{X_i} = E[X(t_i)] = \overline{X}, \ \forall\, i = 1, 2, \ldots, N$: constant

(ii) $C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$: function of time difference only

$\Longrightarrow$ The covariance matrix will be a **symmetric matrix !**.
($\because$ since $R_{XX}(\tau)$ is symmetric.)

Example 7.9

A WSS Gaussian r.p. $X(t)$ has the following characteristics:

(i) $\overline{X} = 4$.
(ii) $R_{XX}(\tau) = 25e^{-3|\tau|}$.

Then, what is the p.d.f. of a random vector $(X(t_1), X(t_2), X(t_3))^T$, where $t_i = t_0 + \frac{1}{2}(i-1), \ \ i = 1, 2, 3$ for an arbitrary t_0?

Solution:

We only need to find the mean vector and the covariance matrix !

(1) $\underline{\overline{X}} = (4, 4, 4)^T$, since $X(t)$ is WSS.
(2) $C_{XX}(t_i, t_k) = 25e^{-3|t_i - t_k|} - 16$, where $t_i - t_k = \frac{1}{2}(i-k)$ for $i, k = 1, 2, 3$.

$$\mathbf{C}_X = \begin{bmatrix} 9 & 25e^{-\frac{3}{2}} - 16 & 25e^{-3} - 16 \\ 25e^{-\frac{3}{2}} - 16 & 9 & 25e^{-\frac{3}{2}} - 16 \\ 25e^{-3} - 16 & 25e^{-\frac{3}{2}} - 16 & 9 \end{bmatrix} \text{ :symmetric}$$

[21] $C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X_i]E[X_k]$.

[22] If $X(t)$ is WSS, then

$$C_{XX}(t_i, t_k) = R_{XX}(t_i, t_k) - E[X_i]E[X_k] = R_{XX}(t_i - t_k) - \overline{X}^2 \text{ : function of time difference only}$$

.

Definition 7.19 Two random processes $X(t)$ and $Y(t)$ are called *jointly Gaussian* if rnadom variables $X(t_1), X(t_2), \ldots, X(t_N), Y(t_1^{'}), Y(t_2^{'}), \ldots, Y(t_M^{'})$ are jointly Gaussian for any N, $t_1, t_2, \ldots t_N$, and M, $t_1^{'}, t_2^{'}, \ldots, t_M^{'}$.

FACT:

If two r.p.'s $X(t)$ and $Y(t)$ are jointly Gaussian, then:

$$\textbf{statistical independence} \quad \equiv \quad \textbf{uncorrelatedness}$$

Proof: assignment [23]

[23] You only need to prove that *uncorrelatedness* implies the *statistical independence*, since the other direction is always true.