Contents

Chapter 8

Random Processes - Spectral Characteristics

So far, we have considered the characteristics of random processes in time domain, i.e.

$$\begin{cases} \text{autocorrelation function} \\ \text{cross-correlation function} \\ \text{covariance function} \\ \text{atationarity \& ergodicity} \\ \vdots \end{cases}$$

Objective:

Now, we study the spectral characteristics of random processes via "Fourier transform".

Recall: Wiener-Khinchin theorem

$$\text{Auto power spectral density} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{Autocorrelation}$$

$$\text{Cross power spectral density} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \text{Cross-correlation}$$

8.1 Power spectral density (PSD) and its properties

8.1.1 Fourier transform: review

The Fourier transform pair for a non-periodic signal $x(t)$ is as follows:

$$X(\omega) = \mathcal{F}[x(t)] \triangleq \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

(cf) $X(\omega)$ represents the distribution of frequency components contained in the signal $x(t)$!

Note: Condition(s) for the existence of F.T.:

(a) $x(t)$ must be *absolutely integrable*, i.e.

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

(b) $x(t)$ must satisfy the Dirichlet conditions given below:

(i) $x(t)$ must have finite number of finite discontinuities within any finite time interval.

(ii) $x(t)$ must have finite number of finite maxima and minima within any time interval.

$\Longrightarrow$ Often many sample functions ($x(t)$) from a random process $X(t)$ do not satisfy the above condition(s), i.e. the Fourier transform of $X(t)$ does not exist. [1]

$\Longrightarrow$ Instead of direct F.T., we search for the distribution of **power** [2] along the frequency domain in order to guarantee the existence of frequency domain representation of a random process as $T \to \infty$.

$\Longrightarrow$ **Concept of Power Spectral Density (PSD) !!!**

[1]Especially the condition (a), and note that $\int_{-\infty}^{\infty} |x(t)|dt < \infty$ cannot be checked in practice!

[2]This implies that we will consider $X(t)$ within *finite duration* time interval.

8.1.2 Derivation of power spectral density

[3]

Define $x_T(t)$ as a portion of a sample function $x(t)$ from a r.p. $X(t)$:

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ \\ 0 & \text{elsewhere} \end{cases}$$

Then, since $x_T(t)$ is absolutely integrable [4], i.e.

$$\int_{-\infty}^{\infty} |x_T(t)| dt = \int_{-T}^{T} |x(t)| dy < \infty,$$

there ∃ the Fourier transform $X_T(\omega)$ of $x_T(t)$.

$$X_T(\omega) = \int_{-\infty}^{\infty} x_T(t) e^{-j\omega t} dt = \int_{-T}^{T} x(t) e^{-j\omega t} dt$$

Since $x_T(t)$ has its F.T., the Parseval's theorem holds, i.e: the energy(power) in time domain equals to the energy(power) in frequency domain, from which it follows:

$$E = \left(\int_{-\infty}^{\infty} x_T^2(t) dt \right) = \int_{-T}^{T} x^2(t) dt \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 \, d\omega \tag{8.1}$$

Expressing (8.1) in terms of average power in $(-T, T)$, we get: [5]

$$P = \frac{1}{2T} \int_{-T}^{T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(\omega)|^2}{2T} d\omega \tag{8.2}$$

Notes on (8.2):

1. The intgrand of RHS is in the form of power spectral density, i.e. it represents the distribution of power in $x_T(t)$ along frequency.
2. But, only for $-T < t < T$ in $x(t)$, i.e it does not represent the entire sample function $x(t)$ from $X(t)$, and therefore we must let $T \to \infty$.
3. Also, it is only for a specific sample function $x(t)$, i.e. it is a random variable for the r.p. $X(t)$, and therefore we must take the expectation of it!

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[3] Sometimes it called also as the power denisty spectrum.

[4] Assuming $x(t)$ satisfies the Dirichlet conditions.

[5] This corresponds to the power of $x(t)$, or the energy of $x_T(t)$.

Therefore: by taking the mathematical expectation of (8.2) and letting $T \to \infty$, we obtain the average power P_{XX} of the random process $X(t)$.

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[x^2(t)\right] dt \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} d\omega \tag{8.3}$$

Remarks:

(i) Notice that the LHS of (8.3) is the *time average of the 2nd moment*, i.e. $A\{E[X^2(t)]\}$.

(ii) If $x(t)$ is WSS, then $P_{XX} = \overline{X^2} =$ constant [6] .

From (8.3), we have the following definition of the power spectral density for a random process $X(t)$:

$$S_{XX}(\omega) \triangleq \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

Corresponding average power comtained in $X(t)$ can be calculated by:

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$$

[6] If $X(t)$ is wss, $E[X^2(t)] = R_{XX}(0) = \overline{X^2}$.

Example 8.1

Calculate the average power of the following random process $X(t)$ both in time and frequency domains.

$$X(t) = A\cos(\omega_0 t + \Theta)$$

where A, ω_0 are constants, and $\Theta \sim U[0, \frac{\pi}{2}]$.

Solution:

(1) Average power in time domain:

$$P_{XX} = A\left\{E\left[X^2(t)\right]\right\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E\left[X^2(t)\right]dt$$

Here, the second mement of $X(t)$ is as follows:

$$\begin{aligned}
E\left[X^2(t)\right] &= E\left[A^2\cos^2(\omega_0 t + \Theta)\right] \\
&= E\left[\frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega_0 t + 2\Theta)\right] \\
&= \frac{A^2}{2} + \frac{A^2}{2}\int_0^{\frac{\pi}{2}}\cos(2\omega_0 t + 2\theta)\cdot\frac{2}{\pi}d\theta \\
&= \frac{A^2}{2} + \frac{A^2}{2}\frac{1}{2}\{\sin(2\omega_0 t + \pi) - \sin(2\omega_0 t)\}\cdot\frac{2}{\pi} \\
&= \frac{A^2}{2} + \frac{A^2}{2}\frac{1}{2}\{-\sin(2\omega_0 t) - \sin(2\omega_0 t)\}\cdot\frac{2}{\pi} \\
&= \frac{A^2}{2} - \frac{A^2}{\pi}\sin(2\omega_0 t)
\end{aligned}$$

(cf) Note that $E\left[X^2(t)\right] \neq$ constant, which means $X(t)$ is NOT WSS !

The average power of $X(t)$ is then: [7]

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{\frac{A^2}{2} - \frac{A^2}{\pi}\sin(2\omega_0 t)\right\}dt = \frac{A^2}{2}\ \text{(watts)}$$

[7] Note that $\int_{-T}^{T}\sin(2\omega_0 t)dt = 0$, since cosine function is an even function.

(2) Average power in frequency domain (using PSD):

From the definition of the power spectral density, we have:

$$S_{XX}(\omega) \triangleq \lim_{T\to\infty} E\left[|X_T(\omega)|^2\right] \cdot \frac{1}{2T}$$

where the Fourier transform of $x_T(t)$ can be derived as: [8]

$$\begin{aligned} X_T(\omega) &= \int_{-T}^{T} A\cos(\omega_0 t + \Theta) s^{-j\omega t} dt \\ &\qquad \vdots \quad \text{(assignment)} \\ &= AT\left\{\text{Sa}\left[(\omega-\omega_0)T\right]e^{j\Theta} + \text{Sa}\left[(\omega+\omega_0)T\right]e^{-j\Theta}\right\} \end{aligned}$$

where $\text{Sa}(x) \triangleq \frac{\sin(x)}{x}$.

Now,

$$\begin{aligned} |X_T(\omega)|^2 &= X_T(\omega)\cdot X_T^*(\omega) \\ &= A^2T^2\Big\{\text{Sa}^2\left[(\omega-\omega_0)T\right] + \text{Sa}\left[(\omega-\omega_0)T\right]\text{Sa}\left[(\omega+\omega_0)T\right]e^{j2\Theta} \\ &\quad + \text{Sa}\left[(\omega-\omega_0)T\right]\text{Sa}\left[(\omega+\omega_0)T\right]e^{-j2\Theta} + \text{Sa}^2\left[(\omega+\omega_0)T\right]\Big\} \\ &= A^2T^2\Big\{\text{Sa}^2\left[(\omega-\omega_0)T\right] + \text{Sa}^2\left[(\omega+\omega_0)T\right] \\ &\quad + 2\text{Sa}\left[(\omega-\omega_0)T\right]\text{Sa}\left[(\omega+\omega_0)T\right]\cos(2\Theta)\} \end{aligned}$$

Therefore, we have: [9]

$$E\left[|X_T(\omega)|^2\right] = A^2T^2\left\{\text{Sa}^2\left[(\omega-\omega_0)T\right] + \text{Sa}^2\left[(\omega+\omega_0)T\right]\right\}$$

The power spectral density $S_{XX}(\omega)$ now becomes:

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{A^2T}{2}\left\{\text{Sa}^2\left[(\omega-\omega_0)T\right] + \text{Sa}^2\left[(\omega+\omega_0)T\right]\right\}$$

[8] In the process of derivation, you may have to use the Euler's formula to get the final expression.

[9] Here, notice the fact: $E\left[\cos(2\Theta)\right] = \int_0^{\frac{\pi}{2}} \cos(2\theta)\cdot\frac{2}{\pi}d\theta = \frac{2}{\pi}\frac{1}{2}\left[\sin(2\theta)\right]_0^{\frac{\pi}{2}} = \frac{1}{\pi}\left[\sin(\pi)-\sin(0)\right] = 0.$

Fact: From *Introduction to Random Signals and Communication Theory* by Lathi.

$$\lim_{k\to\infty} \frac{k}{\pi} \,\mathrm{Sa}(kt) = \delta(t)$$

or

$$\lim_{k\to\infty} \frac{k}{\pi} \,\mathrm{Sa}^2(kt) = \delta(t)$$

Figure 8.1: Convergence of the sampling function Sa(·).

Therefore, the power spectral density $S_{XX}(\omega)$ of $X(t)$ can be expressed as follows:

$$\begin{aligned} S_{XX}(\omega) &= \lim_{T\to\infty} \frac{A^2 T}{2} \left\{ \mathrm{Sa}^2\left[(\omega-\omega_0)T\right] + \mathrm{Sa}^2\left[(\omega+\omega_0)T\right] \right\} \\ &= \frac{A^2\pi}{2} \left\{ \delta(\omega-\omega_0) + \delta(\omega+\omega_0) \right\} \end{aligned}$$

Corresponding average power of $X(t)$ is then:

$$\begin{aligned} P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega \\ &= \frac{1}{2\pi} \left\{ \frac{A^2\pi}{2} + \frac{A^2\pi}{2} \right\} \\ &= \frac{A^2}{2} \text{ (watts)} \end{aligned}$$

8.1.3 Properties of PSD

1. $S_{XX}(\omega)$ is real.

2. $S_{XX}(\omega) \geq 0$

3. If $X(t)$ is real, then $S_{XX}(\omega) = S_{XX}(-\omega)$, i.e. even function of ω.

4. The average power of $X(t)$ can be evaluated as:

$$P_{XX} = A\left\{E\left[X^2(t)\right]\right\} \overset{\text{or}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$$

5. The PSD of the derivative of $X(t)$ is as follows:

$$S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega) \quad \text{where } \dot{X}(t) = \frac{d}{dt}X(t)$$

6. If $X(t)$ is real, then the *PSD* and the *time average of correlation function* ate Fourier transform pair:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} A\left[R_{XX}(t, t+\tau)\right] e^{-j\omega\tau} d\tau = \mathcal{F}\left\{A\left[R_{XX}(t, t+\tau)\right]\right\}$$

or

$$A\left[R_{XX}(t, t+\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \mathcal{F}^{-1}\left\{S_{XX}(\omega)\right\}$$

7. If $X(t)$ is at least WSS, then above relation in 6 becomes as follows:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = \mathcal{F}\left\{R_{XX}(\tau)\right\}$$

or

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = \mathcal{F}^{-1}\left\{S_{XX}(\omega)\right\}$$

: called **Wiener Khinchine Theorem**

Proof:

1. $S_{XX}(\omega) \triangleq \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$ = real, since $|X_T(\omega)|$ is real.

2. $S_{XX}(\omega) \triangleq \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} \geq 0$, since $|X_T(\omega)|^2 \geq 0$.

3. Notice that

$$\begin{aligned} X_T(-\omega) &= \int_{-T}^{T} X(t) e^{j\omega t} dt \\ &= \left(\int_{-T}^{T} X^*(t) e^{-j\omega t} dt\right)^* \\ &= \left(\int_{-T}^{T} X(t) e^{-j\omega t} dt\right)^* \quad : \text{since } X(t) \text{ is real} \\ &= X_T^*(\omega) \end{aligned}$$

   Therefore, we have:

$$\begin{aligned} S_{XX}(-\omega) &= \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(-\omega)|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(\omega)|^2\right] \\ &= S_{XX}(\omega) \end{aligned}$$

4. Previously shown.

5. Note that: $\mathcal{F}[\dot{x}_T(t)] = j\omega \mathcal{F}[x_T(t)]\ j\omega X_T(\omega) \triangleq \dot{X}_T(\omega)$, and thus we have:

$$\begin{aligned} S_{\dot{X}\dot{X}}(\omega) &= \lim_{T\to\infty} \frac{1}{2T} E\left[\left|\dot{X}_T(-\omega)\right|^2\right] \\ &= \lim_{T\to\infty} \frac{1}{2T} \omega^2 E\left[|X_T(-\omega)|^2\right] \\ &\triangleq \omega^2\, S_{XX}(\omega) \end{aligned}$$

6. Will be shown in the next section...

8.1.4 RMS bandwidth of the PSD

Suppose (i) $X(t)$ is real, [10] and (ii) $X(t)$ is a lowpass process whose power spectral density is as follows:

Figure 8.2: The PSD of a lowpass random process $X(t)$.

$\Longrightarrow$ Normalize $S_{XX}(\omega)$ with its area to get $\widehat{S}_{XX}(\omega)$, i.e.

$$\widehat{S}_{XX}(\omega) = \frac{S_{XX}(\omega)}{\displaystyle\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega}$$

$\Longrightarrow$ Note that $\widehat{S}_{XX}(\omega)$ is similar to a p.d.f. with its mean=0.

$\Longrightarrow$ We define the RMS bandwidth W_{rms} of $X(t)$ as follows:

$$W_{\text{rms}}^2 \triangleq \int_{-\infty}^{\infty} \omega^2 \widehat{S}_{XX}(\omega)d\omega$$

(cf) This corresponds to the amount of dispersion (i.e variance) in power from the viewpoint of frequency, and notice tha similarity between W_{rms}^2 and σ^2(variance w/ its mean zero).

Likewise, for a real, bandpass process $X(t)$, the RMS bandwidth is defined as:

$$W_{\text{rms}}^2 \triangleq 4 \cdot \int_{0}^{\infty} (\omega - \overline{\omega})^2 \widehat{S}_{XX}(\omega)d\omega$$

where

$$\widehat{S}_{XX}(\omega) \triangleq \frac{S_{XX}(\omega)}{\int_0^{\infty} S_{XX}(\omega)d\omega}, \quad \omega > 0 \quad \text{and} \quad \overline{\omega} \triangleq \int_0^{\infty} \omega\, \widehat{S}_{XX}(\omega)d\omega$$

[10] This means $S_{XX}(\omega)$ is a even function of ω.

Remark: Why the factor of 4? $(\because\ BW_{\text{BP}} = 2\, BW_{\text{LP}})$

(1) Lowpass process;

Figure 8.3: The PSD of a lowpass random process $X(t)$.

$$W_{\text{rms}}^2 \triangleq \int_{-\infty}^{\infty} \omega^2 \widehat{S}_{XX}(\omega) d\omega$$

(2) Bandpass process:

Figure 8.4: The PSD of a bandpass random process $X(t)$.

Considering only for $\omega > 0$,

$$W^2 = \int_0^{\infty} (\omega - \overline{\omega})^2 \widehat{S}_{XX}(\omega) d\omega \quad \text{where} \quad \widehat{S}_{XX}(\omega) = \frac{S_{XX}(\omega)}{\int_0^{\infty} S_{XX}(\omega) d\omega}$$

Therefore, the RMS bandwidth of a bandpass random process W_{rms} is then:

$$W_{\text{rms}}^2 = (2W)^2 = 4W^2 = 4 \cdot \int_0^{\infty} (\omega - \overline{\omega})^2 \widehat{S}_{XX}(\omega) d\omega$$

8.2 Relationship b/w PSD and autcorrelation function

: Proof of properties 6 and 7 in the previous section

Property 6: **Time average of autocorrelation function and the PSD are Fourier transform pair for a real r.p. $X(t)$, i.e.**

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A\left[R_{XX}(t,t+\tau)\right] \tag{8.4}$$

or

$$\mathcal{F}\left\{A\left[R_{XX}(t,t+\tau)\right]\right\} = S_{XX}(\omega)$$

Proof:

$$\begin{aligned}
S_{XX}(\omega) &\triangleq \lim_{T\to\infty}\frac{1}{2T}E\left[|X_T(\omega)|^2\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}E\left[X_T^*(\omega)X_T(\omega)\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}E\left[\int_{-T}^{T}X(t_1)e^{j\omega t_1}dt_1\int_{-T}^{T}X(t_2)e^{-j\omega t_2}dt_2\right] \quad : X(t) \text{ is real} \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}E\left[X(t_1)X(t_2)\right]e^{-j\omega(t_2-t_1)}dt_1dt_2 \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}R_{XX}(t_1,t_2)e^{-j\omega(t_2-t_1)}dt_1dt_2 \\
&\quad (\text{Let } t_1 = t \text{ and } t_2 = t_1+\tau = t+\tau, \text{ then } dt_1 = dt \text{ and } dt_2 = d\tau) \\
&= \lim_{T\to\infty}\frac{1}{2T}\left[\int_{-t-T}^{-t+T}\left\{\int_{-T}^{T}R_{XX}(t,t+\tau)dt\right\}e^{-j\omega\tau}d\tau\right] \\
&= \int_{-\infty}^{\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{XX}(t,t+\tau)dt\right]e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty}A\left[R_{XX}(t,t+\tau)\right]e^{-j\omega\tau}d\tau \\
&\triangleq \mathcal{F}\left\{A\left[R_{XX}(t,t+\tau)\right]\right\}
\end{aligned}$$

Note: If $X(t)$ is a WSS random process, then:

$$\begin{aligned} A\left[R_{XX}(t,t+\tau)\right] &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{XX}(t,t+\tau)dt \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R_{XX}(\tau)dt \\ &= R_{XX}(\tau) \end{aligned}$$

Therefore, (8.4) becomes:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}dt$$

or

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$$

i.e., we have the following relationship between the autocorrelation function and the power spectral density, which is called **Wiener-Khinchine Relation**:

$$R_{XX}(\tau) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad S_{XX}(\omega)$$

Note: Given tha PSD of a random process, we can recover:

(i) The autocorrelation function $R_{XX}(\tau)$ if $X(t)$ is at least WSS.

(ii) The time average of the autocorrelation function $A\left[R_{XX}(t,t+\tau)\right]$, if $X(t)$ is non-stationay.

Self study: Example 7.2-1 of the textbook

8.3 Cross power spectral density and its properties

8.3.1 Cross power spectral density

Given two *real* random processes $X(t)$ and $Y(t)$, define $x_T(t)$ and $y_T(t)$ as portions of sample functions $x(t)$ and $y(t)$ from the r.p.'s $X(t)$ and $Y(t)$, i.e.:

$$x_T(t) \triangleq \begin{cases} x(t) & -T < t < T \\ 0 & \text{elsewhere} \end{cases}$$

$$y_T(t) \triangleq \begin{cases} y(t) & -T < t < T \\ 0 & \text{elsewhere} \end{cases}$$

Then, since $x_T(t)$ and $y_T(t)$ are absolutely integrable, $\exists$ F.T. of them. (i.e., $x_T(t) \overset{\mathcal{F}}{\leftrightarrow} X_T(\omega)$, and $y_T(t) \overset{\mathcal{F}}{\leftrightarrow} Y_T(\omega)$.)

$\Longrightarrow$ The cross power of $x(t)$ and $y(t)$ wothin $[-T, T]$ is:

$$\begin{aligned} P_{XY}(T) &\triangleq \frac{1}{2T}\int_{-T}^{T} x_T(t)y_T(t)dt \\ &= \frac{1}{2T}\int_{-T}^{T} x(t)y(t)dt \\ &\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2T}\left\{X_T^*(\omega)Y_T(\omega)\right\} d\omega \quad : \text{by Parseval's theorem} \end{aligned}$$

: random variable: depending on particular sample functions

$\Longrightarrow$ The average cross power of $x(t)$ and $y(t)$ within $[-T, T]$ is then:

$$\begin{aligned} \overline{P}_{XY}(T) \triangleq E\left[P_{XY}(T)\right] &= \frac{1}{2T}\int_{-T}^{T} E\left[X(t)Y(t)\right] dt \\ &= \frac{1}{2T}\int_{-T}^{T} R_{XY}(t,t)dt \\ &\overset{\text{or}}{\equiv} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2T} E\left[X_T^*(\omega)Y_T(\omega)\right] d\omega \end{aligned}$$

$\Longrightarrow$ The total(overall) average cross power is then, by $T \to \infty$:

$$\begin{aligned} P_{XY} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t,t)dt \\ &\overset{\text{or}}{\equiv} \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega)Y_T(\omega)\right] d\omega \end{aligned}$$

(cf) Notice that the integrand in the second integral above is in the form of the cross power spectral density between $X(t)$ and $Y(t)$, $S_{XY}(\omega)$:

Definition of the cross PSD:

$$S_{XY}(\omega) \triangleq \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega)Y_T(\omega)\right]$$

and corresponding cross power in $X(t)$ and $Y(t)$ is given by:

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)d\omega$$

Likewise, we can define the cross PSD b/w $Y(t)$ and $X(t)$ as:

$$S_{YX}(\omega) \triangleq \lim_{T\to\infty} \frac{1}{2T} E\left[Y_T^*(\omega)X_T(\omega)\right]$$

and corresponding cross power in $X(t)$ and $Y(t)$ is given by:

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)d\omega$$

8.3.2 Properties of cross PSD

Suppose $X(t)$ and $Y(t)$ are *real* random processes, then:

1. Cross PSD is *conjugate symmetric:* [11]

$$S_{XY}(\omega = S_{YX}(-\omega) = S^*_{YX}(\omega)$$

2. The real part of cross PSD is even function of ω:

$$\mathrm{Re}\left[S_{XY}(\omega)\right] = \mathrm{Re}\left[S_{XY}(-\omega)\right]$$
$$\mathrm{Re}\left[S_{YX}(\omega)\right] = \mathrm{Re}\left[S_{YX}(-\omega)\right]$$

3. The imaginary part of cross PSD is odd function of ω:

$$\mathrm{Im}\left[S_{XY}(\omega)\right] = -\mathrm{Im}\left[S_{XY}(-\omega)\right]$$
$$\mathrm{Im}\left[S_{YX}(\omega)\right] = -\mathrm{Im}\left[S_{YX}(-\omega)\right]$$

4. If $X(t)$ and $Y(t)$ are *orthogonal*, [12] then:

$$S_{XY}(\omega) = S_{YX}(\omega) = 0$$

5. If $X(t)$ and $Y(t)$ are *uncorrelated*, [13] and $E[X(t)] = \overline{X}$ and $E[Y(t)] = \overline{Y}$, then:

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi\overline{X}\,\overline{Y}\delta(\omega)$$

6. Cross PSD and the time average of cross correlation function are Fourier transform pair, i.e.:

$$A\left[R_{XY}(t, t+\tau)\right] \overset{\mathcal{F}}{\leftrightarrow} S_{XY}(\omega)$$
$$A\left[R_{YX}(t, t+\tau)\right] \overset{\mathcal{F}}{\leftrightarrow} S_{YX}(\omega)$$

7. If $X(t)$ and $Y(t)$ are JWSS, then:

$$R_{XY}(\tau) \overset{\mathcal{F}}{\leftrightarrow} S_{XY}(\omega)$$
$$R_{YX}(\tau) \overset{\mathcal{F}}{\leftrightarrow} S_{YX}(\omega)$$

proof: Assignment [14]

[11] Recall that the auto PSD $S_{XX}(\omega)$ is always real, and even function of ω.
[12] This means that $E[X(t)Y(t)] = 0$.
[13] This means that $E[X(t)Y(t)] = E[X(t)]E[Y(t)]$.
[14] Similar to those for auto PSD.

Question:
If a real r.p. $W(t)$ is defined as a sum of two real r.p.'s $X(t)$ and $Y(t)$, then what is the PSD of $W(t)$ in terms of the PSD's related to $X(t)$ and $Y(t)$?

Given:

$$W(t) \triangleq X(t) + Y(t)$$

The autocorrelation function of $W(t)$ is:

$$\begin{aligned} R_{WW}(t, t+\tau) &\triangleq E\left[W(t)W(t+\tau)\right] \\ &= E\left[\{X(t)+Y(t)\}\{X(t+\tau)+Y(t+\tau)\}\right] \\ &= E\left[X(t)X(t+\tau)\right] + E\left[X(t)Y(t+\tau)\right] \\ &\quad + E\left[Y(t)X(t+\tau)\right] + E\left[Y(t)Y(t+\tau)\right] \\ &\equiv R_{XX}(t,t+\tau) + R_{XY}(t,t+\tau) + R_{YX}(t,t+\tau) + R_{YY}(t,t+\tau) \end{aligned}$$

Then, the auto PSD of $W(t)$ is:

$$\begin{aligned} S_{WW}(\omega) &= \mathcal{F}\left\{A\left[R_{WW}(t,t+\tau)\right]\right\} \\ &= \mathcal{F}\left\{A\left[R_{XX}(t,t+\tau)\right]\right\} + \mathcal{F}\left\{A\left[R_{XY}(t,t+\tau)\right]\right\} \\ &\quad + \mathcal{F}\left\{A\left[R_{YX}(t,t+\tau)\right]\right\} + \mathcal{F}\left\{A\left[R_{YY}(t,t+\tau)\right]\right\} \\ &= S_{XX}(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_{YY}(\omega) \end{aligned}$$

Note:

(1) If $X(t)$ and $Y(t)$ are orthogonal, i.e. $R_{XY}(t,t+\tau) = R_{YX}(t,t+\tau) = 0$, then:

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)$$

(2) If $X(t)$ and $Y(t)$ are uncorrelated to each other and $E[X(t)] = \overline{X}$, $E[Y(t)] = \overline{Y}$, i.e. $R_{XY}(t,t+\tau) = R_{YX}(t,t+\tau) = \overline{X}\,\overline{Y}$, and thus $\mathcal{F}\left\{A\left[R_{XY}(t,t+\tau)\right]\right\} = \mathcal{F}\left\{A\left[R_{YX}(t,t+\tau)\right]\right\} = \mathcal{F}\left\{\overline{X}\,\overline{Y}\right\} = 2\pi\overline{X}\,\overline{Y}\delta(\omega)$, then:

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + 4\pi\overline{X}\,\overline{Y}\delta(\omega)$$

8.4 Relationship between cross PSD and cross-correlation function

Assignment (READ): The proof of properties 6 & 7 in previous section

Example 8.2

Given the cross-correlation function $R_{XY}(t, t+\tau)$ as follows, [15] find the cross power spectral density $S_{XY}(\omega)$ between $X(t)$ and $Y(t)$.

$$R_{XY}(t, t+\tau) = \frac{AB}{2} \{\sin(\omega_0 \tau + \cos(\omega_0(2t+\tau))\}$$

Solution:

$$S_{XY}(\omega) \triangleq \mathcal{F}\{A[R_{XY}(t, t+\tau)]\}$$

$$\begin{aligned}
A[R_{XY}(t,t+\tau)] &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{XY}(t,t+\tau)dt \\
&= \frac{AB}{2}\sin(\omega_0\tau) + \frac{AB}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos(\omega_0(2t+\tau))\, dt \\
&= \frac{AB}{2}\sin(\omega_0\tau) + \frac{AB}{2} \lim_{T\to\infty} \frac{1}{2T} \left[\frac{1}{2\omega_0}\sin(\omega_0(2t+\tau))\right]_{-T}^{T} \\
&= \frac{AB}{2}\sin(\omega_0\tau) \\
&\quad + \frac{AB}{2} \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2\omega_0} \{\sin(\omega_0(2T+\tau)) + \sin(\omega_0(2T-\tau))\} \\
&= \frac{AB}{2}\sin(\omega_0\tau)
\end{aligned}$$

Therefore, the cross PSD $S_{XY}(\omega)$ becomes:

$$S_{XY}(\omega) = \mathcal{F}\left\{\frac{AB}{2}\sin(\omega_0\tau)\right\} = \frac{AB}{2} \cdot j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Corresponding average cross power between $X(t)$ and $Y(t)$ is then: [16]

$$P_{XY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)d\omega = 0$$

[15] Notice that $X(t)$ and $Y(t)$ are NOT JWSS.

[16] If the cross-correlation was $R_{XY}(t, t+\tau) = \frac{AB}{2}\{\cos(\omega_0\tau + \cos(\omega_0(2t+\tau))\}$, then the cross PSD is $S_{XY}(\omega) = \frac{AB}{2}\cdot\pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$ and thus the average cross power is $P_{XY} = \frac{AB}{2}$(watts).

8.5 Some noise definitions and other topics

: Definitions of noise in terms of PSD

8.5.1 White and colored noise

(1) White noise [17]

Definition 8.1 A sample function $n(t)$ of a WSS noise random process $N(t)$ is called a *white noise* if:

$$S_{NN}(\omega) = \frac{N_0}{2} \quad : \text{constant} \; \forall \; \omega$$

Then, corresponding autocorrelation function of a white noise is in the form of the Dirac delta function, i.e. [18]

$$R_{NN}(\tau) = \mathcal{F}^{-1}\left\{S_{NN}(\omega)\right\} = \frac{N_0}{2}\delta(\tau)$$

Figure 8.5: Auto PSD and autocorrelation function of a white noise.

Note:

(i) White noise is physically unrealizable, since any signal cannot have an infinite power:

$$P_{NN} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)d\omega = \infty$$

(ii) Close approximation to the white noise are: (a) lightening phenomenon (b) thermal noise etc..

[17] Why white? : If you add up all of the light frequencies, you end up getting a white light.

[18] Note that $R_{NN}(\cdot)$ is a function of τ only, since $N(t)$ is WSS; and recall that $\delta(t) \overset{\mathcal{F}}{\leftrightarrow} 1$.

(2) Colored noise [19]

Definition 8.2 A noise which is NOT white is called a colored noise.

Example 8.3

Bandlimited white noise

$$S_{NN}(\omega) = \begin{cases} \frac{P\pi}{W} & -W < \omega < W \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{power} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)d\omega = \frac{1}{2\pi}2W \cdot \frac{P\pi}{W} = P\ (\text{watts})$$

Figure 8.6: Auto power spectral density of a colored noise.

Corresponding autocorrelation function is then:

$$R_{NN}(\tau) = \mathcal{F}^{-1}\left\{S_{NN}(\omega\right\} = P \cdot \mathrm{Sa}\left(W\tau\right)$$

where $\mathrm{Sa}(x) \triangleq \frac{\sin(x)}{x}$.

$$\text{power} = R_{NN}(0) = P\ (\text{watts})$$

Figure 8.7: Autocorrelation function of a colored noise.

[19] Why colored? : portions of visible light frequencies give a certain color.

8.5.2 Product device response to a random signal

Figure 8.8: Block diagram of a product device.

Example 8.4

AMSC(amplitude modulation w/ suppressed carrier) [20]

Figure 8.9: AMSC.

Question:

Suppose $X(t)$ is a WSS r.p., then what is the PSD of the output $Y(t)$ in terms of the PSD of $X(t)$?

Steps:

(i) $Y(t)$

(ii) $R_{YY}(t, t+\tau)$

(iii) $A\left[R_{YY}(t, t+\tau)\right]$

(iv) $S_{YY}(\omega) = \mathcal{F}\left\{A\left[R_{YY}(t, t+\tau)\right]\right\}$

[20]Note that the carrier $c(t) = A\cos(\omega_c t)$ is a deterministic signal, i.e. A and ω_c are constant.

(i) The output signal:

$$Y(t) = AX(t)\cos(\omega_c t)$$

(ii) The autocorrelation of the output:

$$\begin{aligned}
R_{YY}(t, t+\tau) &\triangleq E\left[Y(t)Y(t+\tau)\right] \\
&= E\left[AX(t)\cos(\omega_c t)AX(t+\tau)\cos(\omega_c t + \omega_c \tau)\right] \\
&= A^2 R_{XX}(\tau)\cos(\omega_c t)\cos(\omega_c t + \omega_c \tau) \quad (\because \ X(t) \ \text{ is WSS}) \\
&= \frac{A^2}{2} R_{XX}(\tau)\left\{\cos(\omega_c \tau) + \cos(2\omega_c t + \omega_c \tau)\right\}
\end{aligned}$$

NOTE: Notice that $Y(t)$ is NOT WSS, [21] even if $X(t)$ is WSS, and this is because the product device is a non-linear system!!!

(iii) The time average of $R_{YY}(t, t+\tau)$:

$$\begin{aligned}
A\left[R_{YY}(t, t+\tau)\right] &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{A^2}{2} R_{XX}(\tau)\left\{\cos(\omega_c \tau) + \cos(2\omega_c t + \omega_c \tau)\right\} dt \\
&= \frac{A^2}{2} R_{XX}(\tau)\cos(\omega_c \tau) + \lim_{T\to\infty} \frac{1}{2T}(\text{constant})
\end{aligned}$$

[21] R_{YY} depends on t.

(iv) The auto PSD of $Y(t)$:

$$\begin{aligned}
S_{YY}(\omega) &= \mathcal{F}\left\{\frac{A^2}{2}R_{XX}(\tau)\cos(\omega_c\tau)\right\} \\
&= \frac{A^2}{2}\left[\mathcal{F}\left\{R_{XX}(\tau)\right\} * \mathcal{F}\left\{\cos(\omega_c\tau)\right\}\right] \cdot \frac{1}{2\pi} \\
&= \frac{A^2}{4\pi}\left[S_{XX}(\omega) * \left\{\pi\delta(\omega-\omega_c) + \pi\delta(\omega+\omega_c)\right\}\right] \\
&= \frac{A^2}{4}\left[S_{XX}(\omega-\omega_c) + S_{XX}(\omega+\omega_c)\right]
\end{aligned}$$

Figure 8.10: The auto PSD's of the input $X(t)$ and the output $Y(t)$.

Example: Demodulation of AMSC:

$$Z(t) = Y(t)A\cos(\omega_c t)$$

Express $S_{ZZ}(\omega)$ in terms of $S_{XX}(\omega)$.

: Assignment