Contents

Chapter 9

Linear Systems with Random Inputs

So far, we have studied the "characteristics of random signal":

(1) Time domain: correlation functions, mean, etc.

(2) Frequency domain: power spectral density etc.

From now on, we will deal with the "interaction os random signals with linear systems"

9.1 Linear system fundamentals

9.1.1 General linear system:

where $y(t) = L[x(t)]$

Figure 9.1: A general linear system $L[\cdot]$.

Definition 9.1 A system $L[\cdot]$ is called a *linear system* if:

$$y(t) = L\left[\sum_{i=1}^{N} a_i x_i(t)\right] = \sum_{i=1}^{N} a_i L\left[x_i(t)\right] = \sum_{i=1}^{N} a_i y_i(t)$$

where $y_i(t) \triangleq L\left[x_i(t)\right]$ for $i = 1, 2, \ldots, N$ and a_i's are constants.

Due to the sifting property of the Direc delta function, we have for an arbitrary signal $x(t)$ in general:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Therefore, for a linear system, the output signal $y(t)$ can be expressed as:

$$\begin{aligned} y(t) = L\left[x(t)\right] &= L\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} x(\tau)L\left[\delta(t-\tau)\right]d\tau \quad : \text{due to linearity} \\ &= \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau \end{aligned}$$

where $h(t,\tau) \triangleq L\left[\delta(t-\tau)\right]$ is called the *impulse response* of the system $L[\cdot]$.

Remark:

The response of a linear system is completely determined by its impulse response $h(t,\tau)$!!!

9.1.2 Time invariant system:

Definition 9.2 A system $L[\cdot]$ is called a *time invariant* if:

$$L\left[x(t-t_0)\right] = y(t-t_0)$$

where $y(t) \triangleq L\left[x(t)\right]$.

9.1.3 Linear time invariant (LTI) system:

Definition 9.3 A system $L[\cdot]$ is called an *LTI* system if it is both linear and time invariant:

Figure 9.2: An LTI system.

For an LTI system, let:

$$h(t) \triangleq h(t,0) = L\left[\delta(t-0)\right] = L\left[\delta(t)\right]$$

Then,

$$\begin{aligned} h(t,\tau) &= L\left[\delta(t-\tau)\right] \\ &= L\left[\delta(t)\right]_{t\to t-\tau} \quad (\because \text{ time invariant}) \\ &= h(t-\tau) \end{aligned}$$

Therefore, the I/O realtionship of an LTI system becomes: [1]

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t)$$

: convolution integral

[1]Note: $x(t) * h(t) = h(t) * x(t)$.

9.1.4 Transfer function:

: system characteristic in frequency domain which is equivalent to the impulse response $h(t)$ in time domain [2]

Figure 9.3: An LTI system with $h(t)$ and $H(\omega)$.

From the output signal $y(t)$ expressed in the convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Take the Fourier transform of both sides:

$$\begin{aligned}
Y(\omega) \triangleq \mathcal{F}\{y(t)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right] e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right] d\tau \\
&\quad (\text{let } t-\tau = t') \\
&= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t')e^{-j\omega t'}dt'\right] e^{-j\omega\tau}d\tau \\
&\triangleq H(\omega)\cdot\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau \\
&= H(\omega)X(\omega)
\end{aligned}$$

i.e.:

$$y(t) = h(t) * x(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad Y(\omega) = H(\omega)X(\omega)$$

Definition 9.4 The Fourier transform of the impulse response for an LTI system is called the *transfer function*:

$$H(\omega) \triangleq \mathcal{F}\{h(t)\}$$

[2]Another way of definition: if $x(t) = e^{j\omega t}$, then $H(\omega) \triangleq \frac{L[e^{j\omega t}]}{e^{j\omega t}} = \frac{y(t)}{x(t)}$.

9.1.5 Idealized systems:

The transfer function of an idealized system *in a practical sense* is in the following form:

(i) Magnitude: flat with unit gain

(ii) Phase : linear phase

Figure 9.4: An example of an ideal LPF: (1) practical, (2) theoretical.

Note: The linear phase is needed for the "*distortionless*" output of the system:

Figure 9.5: An LTI system.

From the I/O relationship of:

$$Y(\omega) = H(\omega)X(\omega)$$

we have:

$$\begin{aligned} |Y(\omega)| e^{j\Phi_Y(\omega)} &= |H(\omega)| e^{j\Phi_H(\omega)} \cdot |X(\omega)| e^{j\Phi_X(\omega)} \\ &= |H(\omega)| |X(\omega)| e^{j[\Phi_H(\omega)+\Phi_X(\omega)]} \end{aligned}$$

In words, the magnitude characteristic of the system works in a *multiplicative* way, whereas the phase characteristic of the system works in an *additive* way.

Example 9.1

Let the input of the system be as follows:

$$x(t) = \sin(\omega_0 t + \theta)$$

Then, the output will be:

$$\begin{aligned} y(t) &= \sin\left(\omega_0 t + \theta + \Phi_H(\omega_0)\right) \\ &= \sin\left(\omega_0 t + \theta + (-\alpha\omega_0)\right) \\ &= \sin\left(\omega_0 (t-\alpha) + \theta\right) \\ &= x(t-\alpha) \end{aligned}$$

Notice that the output $y(t)$ is just a shifted version of the input $x(t)$!!!

(cf) If $\Phi_H(\omega)$ were not linear, some distortions in $y(t)$ would have occurred.

9.1.6 Causal and stable systems:

Definition 9.5 An LTI system is called *causal* if:

$$y(t_0) = f\left[x(t)\right], \quad \text{where } t \leq t + 0$$

Fact: If the impulse response $h(t)$ of an LTI system satisfies $h(t) = 0, \ \forall\, t < 0$, then the system is a causal system.

Definition 9.6 A bounded input/bounded output LTI system is called a *stable* system.

Fact: The impulse response $h(t)$ of a stable LTI system should satisfy:

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

9.2 Random signal response of linear systems

: Response of a *stable* LTI system to a r.p. $X(t)$

Figure 9.6: A stable LTI system with random input.

Objective: Characteristics of the output $Y(t)$ [3]

(i) Time domain: mean, variance, correlation functions etc.

(ii) Freq. domain: power spectral density etc.

Question: [4] If $X(t)$ is WSS, then (1) is $Y(t)$ WSS? (2) are $X(t)$ and $Y(t)$ JWSS?

9.2.1 System response

Since the output of an LTI system is the convolution integral between the impulse response and the input, we have:

$$\begin{aligned} Y(t) &= h(t) * X(t) \\ &= \int_{-\infty}^{\infty} h(\tau) X(t-\tau) d\tau \end{aligned}$$

or

$$\begin{aligned} Y(t) &= X(t) * h(t) \\ &= \int_{-\infty}^{\infty} X(\tau) h(t-\tau) d\tau \end{aligned}$$

[3] Note that these are the criteria used for characterizing a random process.

[4] Recall that the output process of a non-linear system (e.g. product device) is not WSS, even though the input is WSS.

9.2.2 Mean and mean squared value of the response

Assuming $X(t)$ is a WSS process, then:

(i) Mean:

$$
\begin{aligned}
E[Y(t)] &= E\left[\int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} h(\tau)E[X(t-\tau)]\,d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\cdot\overline{X}d\tau \\
&= \overline{X}\int_{-\infty}^{\infty} h(\tau)d\tau \quad : \text{independent of } t \\
& \quad \text{(integral is constant, since the system is stable)} \\
&\triangleq \overline{Y} \quad \text{:constant}
\end{aligned}
\tag{9.1}
$$

(ii) Mean squared value:

$$
\begin{aligned}
E\left[Y^2(t)\right] &= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1\int_{-\infty}^{\infty} h(\tau_2)X(t-\tau_2)d\tau_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E[X(t-\tau_1)X(t-\tau_2)]\,d\tau_1d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)R_{XX}(\tau_1-\tau_2)d\tau_1d\tau_2 \\
&\triangleq \overline{Y^2} \quad : \text{independent of } t
\end{aligned}
$$

(cf) Variance(2nd sentral moment) : $\sigma_Y^2(t) = \overline{Y^2} - \overline{Y}^2 \triangleq \sigma_Y^2$ (constant).

9.2.3 Autocorrelation function of $Y(t)$

Assuming $X(t)$ is WSS, the autocorrelation of the output $Y(t)$ is given by:

$$
\begin{aligned}
R_{YY}(t, t+\tau) &\triangleq E\left[Y(t)Y(t+\tau)\right] \\
&= E\left[\int_{-\infty}^{\infty} h(\tau_1)X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)X(t+\tau-\tau_2)d\tau_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E\left[X(t-\tau_1)X(t+\tau-\tau_2)\right]d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2 \\
&\quad : \text{independent of } t \text{ (i.e. dependes only on } \tau)
\end{aligned}
$$

i.e.

$$
R_{YY}(t, t+\tau) = R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2 \tag{9.2}
$$

Note:

(1) $Y(t)$ is WSS if $X(t)$ is WSS. ($\because$ from (9.1) and (9.2).)

(2) $R_{YY}(\tau)$ is in the form of two-fold convolution:

$$
R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau) \tag{9.3}
$$

proof:

$$\begin{aligned}
R_{YY}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty} h(\tau_1)\left[\int_{-\infty}^{\infty} h(\tau_2)R_{XX}(\tau+\tau_1-\tau_2)d\tau_2\right]d\tau_1 \\
&= \int_{-\infty}^{\infty} h(\tau_1)\left[h(\tau+\tau_1) * R_{XX}(\tau+\tau_1)\right]d\tau_1 \\
& \quad (\text{let } \tau+\tau_1 = \tau^{'}) \\
&= \int_{-\infty}^{\infty} h(\tau^{'}-\tau)\left[h(\tau^{'}) * R_{XX}(\tau^{'})\right]d\tau^{'} \\
& \quad (\text{ let } h(\tau^{'}) * R_{XX}(\tau^{'}) = y(\tau^{'})) \\
&= \int_{-\infty}^{\infty} y(\tau^{'})h(\tau^{'}-\tau)d\tau^{'} \\
&= \int_{-\infty}^{\infty} y(\tau^{'})h\left(-(\tau-\tau^{'})\right)d\tau^{'} \\
&= y(\tau) * h(-\tau) \\
&= h(\tau) * R_{XX}(\tau) * h(-\tau)
\end{aligned}$$

9.2.4 Cross-correlation between the input and the output

Assuming $X(t)$ is WSS, then

(i) Correlation b/w $X(t)$ and $Y(t)$: [5]

$$\begin{aligned}
R_{XY}(t, t+\tau) &\triangleq E\left[X(t)Y(t+\tau)\right] \\
&= E\left[X(t)\int_{-\infty}^{\infty} h(\tau_1)X(t+\tau-\tau_1)d\tau_1\right] \\
&= \int_{-\infty}^{\infty} h(\tau_1)E\left[X(t)X(t+\tau-\tau_1)\right]d\tau_1 \\
&= \int_{-\infty}^{\infty} h(\tau_1)R_{XX}(\tau-\tau_1)d\tau_1 \\
&= h(\tau) * R_{XX}(\tau) \triangleq R_{XY}(\tau) \qquad (9.4)
\end{aligned}$$

[5] This will be used later for system identification, i.e. estimating $\hat{h}(t)$.

(ii) Similarly, we get the correlation b/w $Y(t)$ and $X(t)$ as:

$$R_{YX}(t.t+\tau) = h(-\tau) * R_{XX}(\tau) \triangleq R_{YX}(\tau) \tag{9.5}$$

Fact:

If $X(t)$ is WSS, then the input $X(t)$ and the output $Y(t)$ of an LTI system are **JWSS**:

(a) $X(t)$ and $Y(t)$ are WSS individually

(b) $R_{XY}(t, t+\tau) = R_{XY}(\tau)$: function of τ only

Note:

From (9.3), (9.4) and (9.5), the autocorrelation of the output process can be represented in either of the following way:

$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau)$$

or

$$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau)$$

9.3 System evaluation using random white noise

: System identification

Objective: Find the impulse response $h(t)$ of an LTI system

Figure 9.7: An LTI system.

Block diagram:

Figure 9.8: Block diagram of system identification.

Analysis:

Let $X(t)$ be approximately *white noise*, i.e.

$$R_{XX}(\tau) \simeq \left(\frac{N_0}{2}\right)\delta(\tau)$$

Then, the cross-correlation between the input and the output of the system is as follows;

$$\begin{aligned} R_{XY}(\tau) &= h(\tau) * R_{XX}(\tau) \\ &= \int_{-\infty}^{\infty} h(\tau_1) R_{XX}(\tau - \tau_1) d\tau_1 \\ &= \int_{-\infty}^{\infty} h(\tau_1) \frac{N_0}{2} \delta(\tau - \tau_1) d\tau_1 \\ &= \frac{N_0}{2} h(\tau) \quad : \text{by sifting property of } \delta(t) \end{aligned}$$

From which we get:

$$h(\tau) \simeq \frac{2}{N_0} R_{XY}(\tau)$$

Therefore, the estimation of the system's impulse response becomes:

$$\widehat{h}(\tau) = \frac{2}{N_0} \widehat{R_{XY}}(\tau) \approx h(\tau)$$

9.4 Spectral characteristics of system response

Given an LTI system, where the input is a WSS r.p. $X(t)$, and the impulse response $h(t)$ of the system is assumed to be *real*:

Figure 9.9: An LTI system.

Then,

(1) The output $Y(t)$ is WSS.

(2) The input $X(t)$ and the output $Y(t)$ are JWSS.

9.4.1 The PSD of output $Y(t)$

Since $Y(t)$ is WSS, we have:

$$\begin{aligned}
S_{YY}(\omega) &= \mathcal{F}\{R_{YY}(\tau)\} \\
&= \int_{-\infty}^{\infty} R_{YY}(\tau) d\tau \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_{XX}(\tau + \tau_1 - \tau_2) d\tau_1 d\tau_2 \right] e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \left[\int_{-\infty}^{\infty} R_{XX}(\tau + \tau_1 - \tau_2) d\tau \right] d\tau_2 d\tau_1 \\
&= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) S_{XX}(\omega) e^{j\omega\tau_1} e^{-j\omega\tau_2} d\tau_2 d\tau_1 \quad \text{(time shift prop of F.T.)} \\
&= \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1 \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2 S_{XX}(\omega) \\
&= H^*(\omega) H(\omega) S_{XX}(\omega) \qquad \text{(since } h(t) \text{ is assumed to be real)} \\
&= |H(\omega)|^2 \cdot S_{XX}(\omega)
\end{aligned}$$

: direct calculation of $S_{YY}(\omega)$ w/o via $R_{YY}(\tau)$

We call $|H(\omega)|^2$ the **power transfer function** of the system:

$$\text{power transfer function} \triangleq |H(\omega)|^2 = H^*(\omega)H(\omega)$$

Corresponding output power of $Y(t)$ can then be calculated using the PSD $S_{XX}(\omega)$ of the input process $X(t)$ as:

$$\begin{aligned} P_{YY} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 \cdot S_{XX}(\omega)d\omega \end{aligned}$$

(cf) Another simpler way of derivation:

$$\begin{aligned} S_{YY}(\omega) &= \mathcal{F}\{R_{YY}(\tau)\} \\ &= \mathcal{F}\{h(\tau) * h(-\tau) * R_{XX}(\tau)\} \\ &= \mathcal{F}\{h(\tau)\} \cdot \mathcal{F}\{h(-\tau)\} \cdot \mathcal{F}\{R_{XX}(\tau)\} \\ &= H(\omega) \cdot H^*(\omega) \cdot S_{XX}(\omega) \\ &= |H(\omega)|^2 \cdot S_{XX}(\omega) \end{aligned}$$

where we have used the following fact:

$$\begin{aligned} \mathcal{F}\{h(-\tau)\} &= \int_{-\infty}^{\infty} h(-\tau)e^{-j\omega\tau}d\tau \\ &= \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt \qquad \text{(by letting } t = -\tau\text{)} \\ &= \left(\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt\right)^* \quad \text{(since } h(t) \text{ is assumed to be real)} \\ &= H^*(\omega) \end{aligned}$$

9.4.2 Cross PSD of the input/output

Since $X(t)$ and $Y(t)$ are JWSS, we have:

(i) Cross PSD of $X(t)$ and $Y(t)$:

$$\begin{aligned}
S_{XY}(\omega) &= \mathcal{F}\{R_{XY}(\tau)\} \\
&= \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau_1)R_{XX}(\tau-\tau_1)d\tau_1\right]e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau_1)\left[\int_{-\infty}^{\infty} R_{XX}(\tau-\tau_1)e^{-j\omega\tau}d\tau\right]d\tau_1 \\
&= \int_{-\infty}^{\infty} h(\tau_1)S_{XX}(\omega)e^{-j\omega\tau_1}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau_1)e^{-j\omega\tau_1}d\tau \cdot S_{XX}(\omega) \\
&= H(\omega)\cdot S_{XX}(\omega)
\end{aligned}$$

(ii) Cross PSD of $Y(t)$ and $X(t)$:

$$\begin{aligned}
S_{YX}(\omega) &= \mathcal{F}\{R_{YX}(\tau)\} \\
&= \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\tau_1)R_{XX}(\tau+\tau_1)d\tau_1\right]e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau_1)\left[\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1)e^{-j\omega\tau}d\tau\right]d\tau_1 \\
&= \int_{-\infty}^{\infty} h(\tau_1)S_{XX}(\omega)e^{j\omega\tau_1}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau_1)e^{j\omega\tau_1}d\tau \cdot S_{XX}(\omega) \\
&= \int_{-\infty}^{\infty} h(\tau_1)e^{-j(-\omega)\tau_1}d\tau \cdot S_{XX}(\omega) \\
&= H(-\omega)\cdot S_{XX}(\omega) \\
&= H^*(\omega)\cdot S_{XX}(\omega) \quad \text{:assuming } h(t) \text{ is real}
\end{aligned}$$

(cf) Another simpler way of derivation:

(i) Cross PSD of $X(t)$ and $Y(t)$:

$$\begin{aligned} S_{XY}(\omega) &= \mathcal{F}\{R_{XY}(\tau)\} \\ &= \mathcal{F}\{h(\tau) * R_{XX}(\tau)\} \\ &= \mathcal{F}\{h(\tau)\} \cdot \mathcal{F}\{R_{XX}(\tau)\} \\ &= H(\omega) \cdot S_{XX}(\omega) \end{aligned}$$

(ii) Cross PSD of $Y(t)$ and $X(t)$:

$$\begin{aligned} S_{YX}(\omega) &= \mathcal{F}\{R_{YX}(\tau)\} \\ &= \mathcal{F}\{h(-\tau) * R_{XX}(\tau)\} \\ &= \mathcal{F}\{h(-\tau)\} \cdot \mathcal{F}\{R_{XX}(\tau)\} \\ &= H^*(\omega) \cdot S_{XX}(\omega) \end{aligned}$$

9.5 Noise bandwith of an LTI system

Consider an LTI system with lowpass characteristics, whose impulse response $h(t)$ is assumed to be *real* :

Figure 9.10: The transfer function of an LTI system(lowpass).

Then,

$$H(-\omega) = \int_{-\infty}^{\infty} h(t)e^{+j\omega t}dt = \left(\int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt\right)^* = H^*(\omega)$$

and therefore:

$$|H(-\omega)|^2 = H(-\omega)H^*(-\omega) = H^*(\omega)H(\omega) = |H(\omega)|^2$$

which means that the power transfer function $|H(\omega)|^2$ is an even function of ω.

We apply a **white noise** as an input to the system, whose power spectral density is as follows:

$$S_{XX}(\omega) = \frac{N_0}{2}$$

Then, the output power becomes:

$$\begin{aligned} P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega)d\omega \\ &= \frac{2}{2\pi}\int_{0}^{\infty} |H(\omega)|^2 \frac{N_0}{2}d\omega \\ &= \frac{N_0}{2\pi}\int_{0}^{\infty} |H(\omega)|^2 d\omega \qquad (9.6) \end{aligned}$$

Now, consider an idealized system which is equivalent to the above system from the viewpoints of:

(i) same output power

(ii) same value of power transfer function at $\omega = 0$, i.e. $|H(0)|^2$.

Figure 9.11: An equivalent idealized system $H_I(\omega)$(lowpass).

Then, the power of $Y(t)$ from the idealized system is:

$$
\begin{aligned}
P_{YY} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H_I(\omega)|^2 \cdot \frac{N_0}{2} d\omega \\
&= \frac{N_0}{4\pi} \cdot 2\int_{0}^{W_N} |H(0)|^2 \, d\omega \\
&= \frac{N_0 \, |H(0)|^2 \, W_N}{2\pi} \qquad (9.7)
\end{aligned}
$$

From (9.6) and (9.7), we get:

$$
\frac{N_0}{2\pi}\int_0^{\infty} |H(\omega)|^2 \, d\omega = \frac{N_0}{2\pi} |H(0)|^2 \, W_N
$$

And the bandwidth of the equivalent idealized system is then:

$$
W_N = \frac{\int_0^{\infty} |H(\omega)|^2 \, d\omega}{|H(0)|^2}
$$

We call W_N the **noise bandwidth** [6] of the system

[6] This term implies the white noise equivalent of the systems's bandwidth.

9.6 Bandpass, bandlimited, and narrowband processes

Definition 9.7 Bandpass process:

A random process $N(t)$ is called *bandpass* if its PSD $S_{NN}(\omega)$ has its significant portion concentrated around $\omega = \omega_0 \neq 0$, i.e.

Figure 9.12: The PSD $S_{NN}(\omega)$ of a typical bandpass random process.

Note:
$S_{NN}(0)$ does not necessarily have to be zero ! It only requires to be a relatively small value compared to $S_{NN}(\omega_0)$.

Definition 9.8 Bandlimited process:

A bandpass random process $N(t)$ is called *bandlimited* if its PSD $S_{NN}(\omega)$ is zero outside of some frequency band of width W concentrated around $\omega = \omega_0 \neq 0$, i.e.

Figure 9.13: The PSD $S_{NN}(\omega)$ of a typical bandlimited random process.

Definition 9.9 Narrowband process:

A bandlimited random process $N(t)$ is called *narrowband* if $\omega_0 \gg W$ in its PSD $S_{NN}(\omega)$, i.e.

Figure 9.14: The PSD $S_{NN}(\omega)$ of a typical narrowband random process.

9.6.1 Typical narrowband random process

Judging from the PSD $S_{NN}(\omega)$ (of a narrowband r.p.), a typical narrowband r.p. should have frequencies near $\omega = \omega_0$, along with relatively *slowly varying amplitude(envelop)* [7] $A(t)$ and *slowly varying phase* [8] $\phi(t)$ as well, i.e.:

$$N(t) = A(t)\cos\left(\omega_0 t + \phi(t)\right) \tag{9.8}$$

where you should be reminded that $A(t)$ and $\phi(t)$ are random processes.

Figure 9.15: A sample function $n(t)$ of a narrowband random process $N(t)$.

[7] This means that $W \ll \omega_0$.

[8] This means that $\omega_0 - \frac{W}{2} < \omega < \omega_0 + \frac{W}{2}$.

Note: (Refer *Davenport and Root*) :may be omitted

(1) If $N(t)$ is Gaussian, then $A(t)$ is Rayleigh and $\phi(t)$ is uniform over $[0, 2\pi]$.

(2) $A(t)$ and $\phi(t)$ are not statistically independent when $N(t)$ is Gaussian.

(3) But, for a fixed $t = t_0$, $A(t_0)$ and $\phi(t_0)$ are independent random variables.

Another way of expressing a narrowband r.p.:

$$\begin{aligned} N(t) &= A(t)\cos(\omega_0 t + \phi(t)) \\ &= A(t)\cos(\omega_0 t)\cos(\phi(t)) - A(t)\sin(\omega_0 t)\sin(\phi(t)) \\ &= A(t)\cos(\phi(t)) \cdot \cos(\omega_0 t) - A(t)\sin(\phi(t)) \cdot \sin(\omega_0 t) \\ &\overset{\text{let}}{=} X(t) \cdot \cos(\omega_0 t) - Y(t) \cdot \sin(\omega_0 t) \end{aligned} \tag{9.9}$$

where

$$X(t) \triangleq A(t)\cos(\phi(t))$$
$$Y(t) \triangleq A(t)\sin(\phi(t))$$

and

$$A(t) = \sqrt{X^2(t) + Y^2(t)}$$
$$\phi(t) = \tan^{-1}\left[\frac{Y(t)}{X(t)}\right]$$

(cf) From now on, we will concentrate on a narrowband r.p. $N(t)$ in the form of (9.9).

9.6.2 Properties of narrowband r.p. $N(t)$

$$N(t) = X(t) \cdot \cos(\omega_0 t) - Y(t) \cdot \sin(\omega_0 t)$$

Suppose $N(t)$ is WSS with following characteristics:

(i) Mean: $E\left[N(t)\right] = 0$

(ii) The PSD:

$$S_{NN}(\omega) = \begin{cases} \text{non-zero}, & 0 < \omega_0 - W_1 < |\omega| < \omega_0 - W_1 + W \\ \text{zero}, & \text{otherwise} \end{cases}$$

Figure 9.16: The PSD of a WSS narrowband random process $N(t)$.

Then, the WSS narrowband r.p. $N(t)$ has the following properties:

property 1: $X(t)$ and $Y(t)$ are JWSS.

property 2: $X(t)$ and $Y(t)$ have zero means:

$$E[X(t)] = E[Y(t)] = 0$$

property 3: $X(t)$, $Y(t)$ and $N(t)$ have equal power:

$$E\left[X^2(t)\right] = E\left[Y^2(t)\right] = E\left[N^2(t)\right]$$

property 4: The autocorrelation of $X(t)$:

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\cos\left((\omega-\omega_0)\tau\right) d\omega$$

property 5: $X(t)$ and $Y(t)$ have the same autocorrelation and PSD:

$$R_{YY}(\tau) = R_{XX}(\tau) \quad \longrightarrow \quad S_{YY}(\omega) = S_{XX}(\omega)$$

property 6: The cross-correlation b/w $X(t)$ and $Y(t)$:

$$R_{XY}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\sin\left((\omega-\omega_0)\tau\right) d\omega$$

property 7: The cross-correlation and PSD b/w $Y(t)$ and $X(t)$: [9]

$$R_{YX}(\tau) = -R_{XY}(\tau) \quad \longrightarrow \quad S_{YX}(\omega) = -S_{XY}(\omega)$$

property 8: $X(t)$ and $Y(t)$ are orthogonal:

$$R_{XY}(0) = E[X(t)Y(t)] = 0, \quad \text{and} \quad R_{YX}(0) = 0$$

property 9: $X(t)$ and $Y(t)$ are lowpass signals: [10]

$$S_{XX}(\omega) = L_p\left[S_{NN}(\omega-\omega_0) + S_{NN}(\omega+\omega_0)\right] = S_{YY}(\omega)$$

property 10: The cross PSD of $X(t)$ and $Y(t)$:

$$S_{XY}(\omega) = jL_p\left[S_{NN}(\omega-\omega_0) - S_{NN}(\omega+\omega_0)\right]$$

[9] Since in general, $R_{XY}(\tau) = R_{YX}(-\tau)$, we can also derive the anti-symmetry of $R_{XY}(\tau)$ as:

$$R_{XY}(\tau) = -R_{XY}(-\tau)$$

.

[10] $L_p(\cdot)$ represents the lowpass portion.

PROOF:

(1) The expectation of $N(t)$ is:

$$E\left[N(t)\right] = E\left[X(t)\right] \cdot \cos(\omega_0 t) - E\left[Y(t)\right] \cdot \sin(\omega_0 t) \equiv 0$$

Therefore, we have:

$$E\left[X(t)\right] = E\left[Y(t)\right] = 0 \quad \textbf{: property 2}$$

(2) Let $W_1 = \frac{W}{2}$ (i.e. ω_0 is at the center of W), and $\omega_0 > \frac{W}{2}$ (i.e. $\exists$ no overlap):

Figure 9.17: The PSD of $N(t)$.

Consider the following system:

Figure 9.18: $N(t)$ through a product device(cosine) and an ideal LPF.

Then, we have the following facts:

(i) $X(t)$ is the output of the above system, i.e.:

$$\begin{aligned} V_1(t) &= 2N(t)\cos(\omega_0 t) \\ &= 2X(t)\cos^2(\omega_0 t) - 2Y(t)\sin(\omega_0 t)\cos(\omega_0 t) \\ &= X(t)\left\{1 + \cos(2\omega_0 t)\right\} - Y(t)\sin(2\omega_0 t) \end{aligned}$$

$$\downarrow \text{LPF}$$

$$X(t)$$

(ii) The autocorrelation of $X(t)$:

$$\begin{aligned}
R_{XX}(t,t+\tau) &= E\left[X(t)X(t+\tau)\right] \\
&= E\left[\int_{-\infty}^{\infty} h(\tau_1)V_1(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)V_1(t+\tau-\tau_2)d\tau_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E\left[V_1(t-\tau_1)V_1(t+\tau-\tau_2)\right]d\tau_1 d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E\left[4N(t-\tau_1)\cos\left(\omega_0(t-\tau_1)\right)\right. \\
&\qquad \left.\cdot N(t+\tau-\tau_2)\cos\left(\omega_0(t+\tau-\tau_2)\right)\right]d\tau_1 d\tau_2 \\
&\quad \text{(since } N(t) \text{ is WSS)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{NN}(\tau+\tau_1-\tau_2) \\
&\qquad \cdot 4\cos\left(\omega_0(t-\tau_1)\right)\cos\left(\omega_0(t+\tau-\tau_2)\right)d\tau_1 d\tau_2 \qquad (9.10)
\end{aligned}$$

Here, we have:

(a) R_{NN} part:

$$R_{NN}(\tau+\tau_1-\tau_2) = \mathcal{F}^{-1}\left\{S_{NN}(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)e^{j\omega(\tau+\tau_1-\tau_2)}d\omega$$

(b) cos part:

$$\begin{aligned}
& 4\cos\left(\omega_0(t-\tau_1)\right)\cos\left(\omega_0(t+\tau-\tau_2)\right) \\
=\ & \left\{e^{j\omega_0(t-\tau_1)} + e^{-j\omega_0(t-\tau_1)}\right\}\left\{e^{j\omega_0(t+\tau-\tau_2)} + e^{-j\omega_0(t+\tau-\tau_2)}\right\} \\
=\ & e^{j\omega_0(2t+\tau-\tau_1-\tau_2)} + e^{-j\omega_0(\tau+\tau_1-\tau_2)} + e^{j\omega_0(\tau+\tau_1-\tau_2)} + e^{-j\omega_0(2t+\tau-\tau_1-\tau_2)} \\
\stackrel{\text{let}}{=}\ & \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)}
\end{aligned}$$

Applying (a) and (b) to (9.10), we get:

[1] First term (I):

$$
\begin{aligned}
&\int\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left[\int_{-\infty}^{\infty}\frac{S_{NN}(\omega)}{2\pi}e^{j\omega(\tau+\tau_1-\tau_2)}d\omega\right]e^{j\omega_0(2t+\tau-\tau_1-\tau_2)}d\tau_1 d\tau_2\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left(\int_{-\infty}^{\infty} h(\tau_1)e^{j(\omega-\omega_0)\tau_1}d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2)e^{-j(\omega+\omega_0)\tau_2}d\tau_2\right)\\
&\qquad\qquad \cdot e^{j2\omega_0 t}e^{j(\omega+\omega_0)\tau}d\omega\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)H^*(\omega-\omega_0)H(\omega+\omega_0)e^{j2\omega_0 t}e^{j(\omega+\omega_0)\tau}d\omega\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left|H(\omega-\omega_0)\right|\left|H(\omega+\omega_0)\right|e^{-j2\alpha\omega_0}e^{j2\omega_0 t}e^{j(\omega+\omega_0)\tau}d\omega\\
&= 0
\end{aligned}
$$

where $H(\omega) = |H(\omega)|\, e^{-j\alpha\omega}$, i.e. $H(\omega)$ has a *linear phase* since it is an ideal LPF, and thus:

$$
\begin{aligned}
H^*(\omega-\omega_0) &= |H(\omega-\omega_0)|\, e^{j\alpha(\omega-\omega_0)}\\
H(\omega+\omega_0) &= |H(\omega+\omega_0)|\, e^{-j\alpha(\omega+\omega_0)}
\end{aligned}
$$

Figure 9.19: $S_{NN}(\omega)$, $|H(\omega-\omega_0)|$ and $|H(\omega+\omega_0)|$.

[2] The fourth term (IV):

In a similar manner, we can show that:

$$
\int\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left[\int_{-\infty}^{\infty}\frac{S_{NN}(\omega)}{2\pi}e^{j\omega(\tau+\tau_1-\tau_2)}d\omega\right]e^{-j\omega_0(2t+\tau-\tau_1-\tau_2)}d\tau_1 d\tau_2 = 0
$$

[3] The second term (II):

$$
\begin{aligned}
& \int\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left[\int_{-\infty}^{\infty} \frac{S_{NN}(\omega)}{2\pi} e^{j\omega(\tau+\tau_1-\tau_2)} d\omega\right] e^{-j\omega_0(\tau+\tau_1-\tau_2)} d\tau_1 d\tau_2 \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left(\int_{-\infty}^{\infty} h(\tau_1)e^{j(\omega-\omega_0)\tau_1} d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2)e^{-j(\omega-\omega_0)\tau_2} d\tau_2\right) \\
& \cdot e^{j(\omega-\omega_0)\tau} d\omega \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left|H(\omega-\omega_0)\right|^2 e^{j(\omega-\omega_0)\tau} d\omega \\
= & \frac{1}{2\pi}\int_{0}^{\infty} S_{NN}(\omega) e^{j(\omega-\omega_0)\tau} d\omega
\end{aligned}
$$

Similarly,

[4] The third term (III):

$$
\int\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left[\int_{-\infty}^{\infty} \frac{S_{NN}(\omega)}{2\pi} e^{j\omega(\tau+\tau_1-\tau_2)} d\omega\right] e^{j\omega_0(\tau+\tau_1-\tau_2)} d\tau_1 d\tau_2
$$

(let $\omega = -\omega'$, then since $S_{NN}(-\omega') = S_{NN}(\omega')$)

$$
\begin{aligned}
= & \int\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\left[\int_{-\infty}^{\infty} \frac{S_{NN}(\omega')}{2\pi} e^{j\omega'(\tau+\tau_1-\tau_2)} d\omega'\right] e^{j\omega_0(\tau+\tau_1-\tau_2)} d\tau_1 d\tau_2 \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)\left|H(\omega-\omega_0)\right|^2 e^{-j(\omega-\omega_0)\tau} d\omega \\
= & \frac{1}{2\pi}\int_{0}^{\infty} S_{NN}(\omega) e^{-j(\omega-\omega_0)\tau} d\omega
\end{aligned}
$$

Therefore, the autocorrelation of $X(t)$ becomes: [11]

$$\begin{aligned}
R_{XX}(t, t+\tau) &= \text{(II)} + \text{(III)} \\
&= \frac{1}{2\pi}\int_0^\infty S_{NN}(\omega) 2\cos\left((\omega-\omega_0)\tau\right) d\omega \\
&= \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\cos\left((\omega-\omega_0)\tau\right) d\omega \quad \textbf{: property 4} \\
&= R_{XX}(\tau) : \text{function of } \tau \text{ only}
\end{aligned}$$

$\Longrightarrow X(t)$ is WSS !

(3) We follow a similar procedure as in (2) for $Y(t)$, i.e. consider the following system:

Figure 9.20: $N(t)$ through a product device(sine) and an ideal LPF.

Then, we have the following facts:

(i) $Y(t)$ is the output of the above system [12], i.e.:

$$\begin{aligned}
V_2(t) &= -2N(t)\sin(\omega_0 t) \\
&= -2X(t)\sin(\omega_0 t)\cos(\omega_0 t) + 2Y(t)\sin^2(\omega_0 t) \\
&= -X(t)\sin(2\omega_0 t) + Y(t)\left\{1-\cos(2\omega_0 t)\right\}
\end{aligned}$$

$$\downarrow \text{LPF}$$

$$Y(t)$$

[11]In fact, note that the terms involving t in (b) resolve to be zero.
[12]Be reminded that $N(t) = X(t)\cos(\omega_0 t) - Y(t)\sin(\omega_0 t)$.

(ii) We repeat the same step in (2)-(ii), to get the autocorrelation of $Y(t)$:

$$\begin{aligned} R_{YY}(t,t+\tau) &= E\left[Y(t)Y(t+\tau)\right] \\ &\quad \vdots \text{ (assignment)} \\ &= \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\cos\left((\omega-\omega_0)\tau\right)d\omega \\ &= R_{YY}(\tau) \\ &\equiv R_{XX}(\tau) \quad \textbf{: property 5} \end{aligned}$$

Consequently, we have:

$$S_{YY}(\omega) = S_{XX}(\omega)$$

$\Longrightarrow$ $Y(t)$ is also WSS !

(4) The PSD $S_{XX}(\omega)$ of $X(t)$:
Since $X(t)$ is WSS, we have:

$$\begin{aligned} S_{XX}(\omega) &= \mathcal{F}\left\{R_{XX}(\tau)\right\} \\ &= \mathcal{F}\left\{\frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\cos\left((\Omega-\omega_0)\tau\right)d\Omega\right\} \quad \text{(by property 4)} \\ &= \int_{-\infty}^{\infty}\left[\frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\cos\left((\Omega-\omega_0)\tau\right)d\Omega\right]e^{-j\omega\tau}d\tau \\ &= \frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\left[\int_{-\infty}^{\infty}\cos\left((\Omega-\omega_0)\tau\right)e^{-j\omega\tau}d\tau\right]d\Omega \\ &= \frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\left\{\pi\delta(\omega-\Omega+\omega_0)+\pi\delta(\omega+\Omega-\omega_0)\right\}d\Omega \\ &= S_{NN}(\omega+\omega_0)+S_{NN}(-\omega+\omega_0) \quad \text{(by sifting property of } \delta(\cdot) \text{)} \\ &= \underbrace{S_{NN}(\omega+\omega_0)}_{(\omega\geq-\omega_0)}+\underbrace{S_{NN}(\omega-\omega_0)}_{(\omega\leq-\omega_0)} \quad \text{(since } \Omega\geq 0 \text{)} \\ &= L_p\left[S_{NN}(\omega-\omega_0)+S_{NN}(\omega+\omega_0)\right] \quad \textbf{: property 9} \end{aligned}$$

Figure 9.21: The auto PSD $S_{XX}(\omega)$ of $X(t)$.

Another way of $S_{XX}(\omega)$ derivation: [13]

First, we find the PSD of $V_1(t)$, and in order to do that determine $R_{V_1V_1}(t, t+\tau)$:

$$\begin{aligned} R_{V_1V_1}(t, t+\tau) &= E\left[V_1(t)V_1(t+\tau)\right] \\ &= E\left[2N(t)\cos(\omega_0 t)2N(t+\tau)\cos(\omega_0 t+\omega_0\tau)\right] \\ &= 4E\left[N(t)N(t+\tau)\right]\cos(\omega_0 t)\cos(\omega_0 t+\omega_0\tau) \\ &= 2R_{NN}(\tau)\left\{\cos(\omega_0\tau)+\cos(2\omega_0 t+\omega_0\tau)\right\} \end{aligned}$$

$$\Longrightarrow\ A\left[R_{V_1V_1}(t, t+\tau)\right] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{V_1V_1}(t, t+\tau)dt = 2R_{NN}(\tau)\cos(\omega_0\tau)$$

: done before when discussing product device

$$\Longrightarrow\ S_{V_1V_1}(\omega) = \mathcal{F}\left\{2R_{NN}(\tau)\cos(\omega_0\tau)\right\} = S_{NN}(\omega-\omega_0)+S_{NN}(\omega+\omega_0)$$

Figure 9.22: The auto PSD $S_{V_1V_1}(\omega)$ of $V_1(t)$.

$\Longrightarrow$ After LPF

$\Longrightarrow\ S_{XX}(\omega) = L_p\left[S_{NN}(\omega-\omega_0)+S_{NN}(\omega+\omega_0)\right]$ **: property 9**.

[13] Refer Ziemer and Tranter.

(5) The power of $X(t)$, $Y(t)$, and $N(t)$:

(i) The average power of $X(t)$:

$$E\left[X^2(t)\right] = R_{XX}(0)$$
$$= \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)d\omega \quad \text{: from property 4}$$

(ii) The average power of $Y(t)$:

$$E\left[Y^2(t)\right] = R_{YY}(0)$$
$$= \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)d\omega \quad \text{: from property 4 \& 5}$$

(iii) The average power of $N(t)$:

$$\begin{aligned} E\left[N^2(t)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)d\omega && \text{: by Parseval's theorem} \\ &= 2\cdot\frac{1}{2\pi}\int_0^{\infty} S_{NN}(\omega)d\omega && (\because\ S_{NN}(\omega) \text{ is symmetric}) \\ &= \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)d\omega \end{aligned}$$

From (i), (ii), and (iii), we have:

$$E\left[X^2(t)\right] = E\left[Y^2(t)\right] = E\left[N^2(t)\right] \quad \textbf{: property 3}$$

(6) The cross-correlation between $X(t)$ and $Y(t)$:

$$
\begin{aligned}
R_{XY}(t,t+\tau) &= E\left[X(t)Y(t+\tau)\right] \\
&= E\left[\int_{-\infty}^{\infty} h(\tau_1)V_1(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)V_2(t+\tau-\tau_2)d\tau_2\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E\left[V_1(t-\tau_1)V_2(t+\tau-\tau_2)\right]d\tau_1 d\tau_2 \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)E\left[4N(t-\tau_1)\cos\left(\omega_0(t-\tau_1)\right)\right. \\
&\qquad \left.\cdot N(t+\tau-\tau_2)\sin\left(\omega_0(t+\tau-\tau_2)\right)\right]d\tau_1 d\tau_2 \\
&\quad \text{(since } N(t) \text{ is WSS)} \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{NN}(\tau+\tau_1-\tau_2) \\
&\qquad \cdot 4\cos\left(\omega_0(t-\tau_1)\right)\sin\left(\omega_0(t+\tau-\tau_2)\right)d\tau_1 d\tau_2 \qquad (9.11)
\end{aligned}
$$

Here, we have:

(a) R_{NN} part:

$$R_{NN}(\tau+\tau_1-\tau_2) = \mathcal{F}^{-1}\left\{S_{NN}(\omega)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{NN}(\omega)e^{j\omega(\tau+\tau_1-\tau_2)}d\omega$$

(b) sinusoidal part:

$$
\begin{aligned}
&4\cos\left(\omega_0(t-\tau_1)\right)\sin\left(\omega_0(t+\tau-\tau_2)\right) \\
=\ & \frac{1}{j}\left\{e^{j\omega_0(t-\tau_1)} + e^{-j\omega_0(t-\tau_1)}\right\}\left\{e^{j\omega_0(t+\tau-\tau_2)} - e^{-j\omega_0(t+\tau-\tau_2)}\right\} \\
=\ & \frac{1}{j}\left\{e^{j\omega_0(2t+\tau-\tau_1-\tau_2)} - e^{-j\omega_0(\tau+\tau_1-\tau_2)} + e^{j\omega_0(\tau+\tau_1-\tau_2)} - e^{-j\omega_0(2t+\tau-\tau_1-\tau_2)}\right\} \\
\equiv\ & \frac{1}{j}\left\{(\mathrm{I}) - (\mathrm{II}) + (\mathrm{III}) - (\mathrm{IV})\right\}
\end{aligned}
$$

We can see that (9.11) is in a similar form of (9.10), except the (-) signs and the $\frac{1}{j}$ scalar !

Therefore, the cross-correlation between $X(t)$ and $Y(t)$ becomes:

$$\begin{aligned}
R_{XY}(t, t+\tau) &= \text{(II)} + \text{(III)} \\
&= \frac{1}{2\pi j}\int_0^\infty S_{NN}(\omega)e^{j(\omega-\omega_0)\tau}d\omega - \frac{1}{2\pi j}\int_0^\infty S_{NN}(\omega)e^{-j(\omega-\omega_0)\tau}d\omega \\
&= \frac{-j}{2\pi}\int_0^\infty S_{NN}(\omega)\cdot\left\{e^{j(\omega-\omega_0)\tau} - e^{-j(\omega-\omega_0)\tau}\right\}d\omega \\
&= \frac{-j}{2\pi}\int_0^\infty S_{NN}(\omega)2j\sin\left((\omega-\omega_0)\tau\right)d\omega \\
&= \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\sin\left((\omega-\omega_0)\tau\right)d\omega \qquad \textbf{: property 6} \\
&= R_{XY}(\tau) : \text{function of } \tau \text{ only}
\end{aligned}$$

$\Longrightarrow$ $X(t)$ and $Y(t)$ are WSS individually.

$\Longrightarrow$ $X(t)$ and $Y(t)$ are JWSS by property 6 **: property 1**

Also, we have:

$$R_{XY}(0) \triangleq E\left[X(t)Y(t)\right] = \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\sin(0)d\omega = 0 \quad \textbf{: property 8}$$

i.e. $X(t)$ and $Y(t)$ are orthogonal !

(7) Cross PSD $S_{XY}(\omega)$ of $X(t)$ and $Y(t)$:

Since $X(t)$ and $Y(t)$ are JWSS, we have:

$$
\begin{aligned}
S_{XY}(\omega) &= \mathcal{F}\{R_{XY}(\tau)\} \\
&= \mathcal{F}\left\{\frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\sin\left((\Omega-\omega_0)\tau\right) d\Omega\right\} \quad \text{(by property 6)} \\
&= \int_{-\infty}^{\infty}\left[\frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\sin\left((\Omega-\omega_0)\tau\right) d\Omega\right] e^{-j\omega\tau} d\tau \\
&= \frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\left[\int_{-\infty}^{\infty}\sin\left((\Omega-\omega_0)\tau\right) e^{-j\omega\tau} d\tau\right] d\Omega \\
&= \frac{1}{\pi}\int_0^\infty S_{NN}(\Omega)\left\{-j\pi\delta(\omega-\Omega+\omega_0)+j\pi\delta(\omega+\Omega-\omega_0)\right\} d\Omega \\
&= j\left\{-S_{NN}(\omega+\omega_0)+S_{NN}(-\omega+\omega_0)\right\} \quad \text{(by sifting property of } \delta(\cdot)) \\
&= \underbrace{-jS_{NN}(\omega+\omega_0)}_{(\omega\geq-\omega_0)}+\underbrace{jS_{NN}(\omega-\omega_0)}_{(\omega\leq-\omega_0)} \quad \text{(since } \Omega\geq 0 \text{)} \\
&= jL_p\left[S_{NN}(\omega-\omega_0)-S_{NN}(\omega+\omega_0)\right] \quad \textbf{: property 10}
\end{aligned}
$$

Figure 9.23: The cross PSD $S_{XY}(\omega)$ of $X(t)$ and $Y(t)$.

Note: If $S_{NN}(\omega)$ is symmetric about $\omega = \omega_0$, then $S_{XY}(\omega) = 0$.

(8) The auto-correlation of $N(t)$:

$$
\begin{aligned}
& R_{NN}(t, t+\tau) \\
\triangleq\ & E\left[N(t)N(t+\tau)\right] \\
=\ & E\left[\{X(t)\cos(\omega_0 t) - Y(t)\sin(\omega_0 t)\}\right. \\
& \qquad \left.\cdot \{X(t+\tau)\cos\left(\omega_0(t+\tau)\right) - Y(t+\tau)\sin\left(\omega_0(t+\tau)\right)\}\right] \\
=\ & R_{XX}(\tau)\cos(\omega_0 t)\cos\left(\omega_0(t+\tau)\right) - R_{YX}(\tau)\sin(\omega_0 t)\cos\left(\omega_0(t+\tau)\right) \\
& -R_{XY}(\tau)\cos(\omega_0 t)\sin\left(\omega_0(t+\tau)\right) + R_{YY}(\tau)\sin(\omega_0 t)\sin\left(\omega_0(t+\tau)\right) \\
& \text{(since } X(t) \text{ and } Y(t) \text{ are JWSS)} \\
=\ & \frac{1}{2}\left\{\cos(2\omega_0 t + \omega_0\tau) + \cos(\omega_0\tau)\right\} R_{XX}(\tau) \\
& -\frac{1}{2}\left\{\sin(2\omega_0 t + \omega_0\tau) - \sin(\omega_0\tau)\right\} R_{YX}(\tau) \\
& -\frac{1}{2}\left\{\sin(2\omega_0 t + \omega_0\tau) + \sin(\omega_0\tau)\right\} R_{XY}(\tau) \\
& +\frac{1}{2}\left\{-\cos(2\omega_0 t + \omega_0\tau) + \cos(\omega_0\tau)\right\} R_{YY}(\tau) \\
=\ & \frac{1}{2}\left[R_{XX}(\tau) - R_{YY}(\tau)\right]\cos(2\omega_0 t + \omega_0\tau) \\
& -\frac{1}{2}\left[R_{XY}(\tau) + R_{YX}(\tau)\right]\sin(2\omega_0 t + \omega_0\tau) \\
& +\frac{1}{2}\left[R_{XX}(\tau) + R_{YY}(\tau)\right]\cos(\omega_0\tau) \\
& +\frac{1}{2}\left[R_{YX}(\tau) - R_{XY}(\tau)\right]\sin(\omega_0\tau) \\
\equiv\ & R_{NN}(\tau) \quad : \text{function of } \tau \text{ only}
\end{aligned}
$$

(cf) Note that $R_{NN}(t, t+\tau) \equiv R_{NN}(\tau)$, since $N(t)$ is WSS !!!

Since R_{NN} should be a function of τ only, we have from the above that:

(i) $R_{XX}(\tau) - R_{YY}(\tau) \equiv 0 \implies R_{XX}(\tau) = R_{YY}(\tau)$: shown before

(ii) $R_{XY}(\tau) + R_{YX}(\tau) \equiv 0 \implies R_{YX}(\tau) = -R_{XY}(\tau)$ **: property 7**

Also, by taking the Fourier transform of (ii), we have:

$$S_{YX}(\omega) = -S_{XY}(\omega)$$

(cf) Relavant properties:

(a) Since $R_{YX}(\tau) = R_{XY}(-\tau)$, which is the general property of the cross-correlation function, the property 7 can be modified to show:

$$\begin{aligned} R_{XY}(\tau) &= -R_{YX}(\tau) \quad \text{(property 7)} \\ &= -R_{XY}(-\tau) \end{aligned}$$

i.e. $R_{XY}(\tau) = -R_{XY}(-\tau)$, which is the anti-symmetry property of the cross-correlation function.

(b) Since $R_{XY}(0) = 0$ by the property 8, we have:

$$R_{YX}(0) = -R_{XY}(0) = 0$$