Contents

Chapter 2

Discrete-Time Signals and Systems

2.1 Review(Introduction)

Mathematical representation of signals:

A function of one or more independent variables

(cf) In most cases, the independent variable is *time*, but not necessarily so; e.g. $v(t)$ is the voltage, $I(x, y)$ is the intensity or brightness of an image, and $I(x, y, t)$ is a moving image.

Category of signals

1. Continuous-time signal(Analog signal): time and amplitude are continuous

   $\downarrow$ *(sampling)*

2. **Discrete-time signal(Sequence)**: time is discrete and amplitude is continuous

   $\downarrow$ *(quantization)*

3. Digital signal : time and amplitude are discrete

(cf): Quantization is for representing signal amplitude with finite number of bits!

$\Rightarrow$ Main focus will be on the discrete-time signals & systems, and quantization effect for digital signal will be dealt with separately...

2.2 Discrete-time signals(Sequences)

Representation: $x = \{x[n]\}$ where n is an integer

$x[n]$ is called the "*n-th*" sample

(cf) We consider only the periodic sampling of analog signals:

$$x[n] = x_a(nT_s)$$

Figure 2.1: Periodic sampling of an analog signal.

where T_s is called the sampling period in (sec), and $\omega_s = \frac{2\pi}{T_s}$ is the sampling frequency in (rad/sec).

Basic sequence operations for DSP:

1. Sum: $z[n] = x[n] + y[n]$
2. Product: $z[n] = x[n]y[n]$
3. Scalar multiple: $y[n] = \alpha x[n]$
4. Time shift: $y[n] = x[n - n_0]$

Note: Sum and product should be carried out *sample by sample.*

Typical(Basic) discrete-time signals:

(1) Unit sample sequence(impulse or Kronecker-delta function):

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

(cf) $\delta[n]$ corresponds to the unit impulse(or Dirac delta) function $\delta(t)$ of continuous-time signals!!!

Figure 2.2: Unit sample sequence.

Remark: Any discrete-time signal $x[n]$ can be represented as a linear combination of scaled, delayed impulses, i.e.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

where $x[k]$ denotes the n-th sample x_k of $x[n]$.

Figure 2.3: A discrete-time signal expressed via $\delta[n]$.

(2) Unit step sequence:

$$u[n] \triangleq \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Figure 2.4: Unit step sequence.

Remark: Relatioship between $\delta[n]$ and $u[n]$: [1]

1. $u[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
2. $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$
3. $\delta[n] = u[n] - u[n-1]$ (: called *backward difference*)

(3) Exponential and sinusoidal sequences:

1. Exponential sequence:

$$\begin{aligned} x[n] &= A\alpha^n \\ &= |A||\alpha|^n e^{j(\omega_0 n + \phi)} \\ &= |A||\alpha|^n \cos(\omega_0 n + \phi) + j|A||\alpha|^n \sin(\omega_0 n + \phi) \\ &\triangleq \text{Re}\{x[n]\} + j\text{Im}\{x[n]\} \end{aligned}$$

where A and α are complex constants, in general, i.e. $A = |A|e^{j\phi}$ and $\alpha = |\alpha|e^{j\omega_0}$.

There exist three different cases for $x[n]$ depending on the value of α:

(a) $|\alpha| > 1$: $x[n]$ oscillates w/ exponentially growing envelope.
(b) $|\alpha| < 1$: $x[n]$ oscillates w/ exponentially decaying envelope.
(c) $|\alpha| = 1$: $x[n]$ oscillates w/ constant envelope.

(cf) For the case of $|\alpha| = 1$, $x[n]$ is called the *"complex exponential sequence"*.

Note:

(a) If A and α are *real*, $x[n] = A\alpha^n$ is *real* as well, and depending on the value of $\alpha \ni$: $0 < \alpha < 1$, $-1 < \alpha < 0$, & $|\alpha| = 1$, $x[n]$ becomes exponentially growing, exponentially decaying, and constant respectively.
(b) We concentrate on *"complex exponential sequence"*, i.e. for the case of $|\alpha| = 1$.

$$\begin{aligned} x[n] &= |A|| \cos(\omega_0 n + \phi) + j|A| \sin(\omega_0 n + \phi) \\ &= |A|e^{j(\omega_0 n + \phi)} \end{aligned}$$

[1] Its analogy for continuous cases are: $u(t) = \int_{-\infty}^{t} \delta(\theta)d\theta$, and $\delta(t) = \frac{du(t)}{dt}$ respectively.

2. Sinusoidal sequence:

$$x[n] = A\cos(\omega_0 n + \phi), \qquad \text{or} \qquad A\sin(\omega_0 n + \phi)$$

Note: Sinusoidal sequences are the real and/or imaginary parts of the *"complex exponential sequence"*.

3. Comparison b/w continuous and discrete-time cases: [2]

   (a) Frequency dimension:

$$\begin{cases} \omega_0 : & \text{frequency} & : \dim(\omega_0) = \text{radian} \\ \phi : & \text{phase} & : \dim(\phi) = \text{radian} \end{cases}$$

   Note the argument of sinusoidal functions is *phase* or *angle* in radians!!!
   $x(t) = A\cos(\omega_0 t + \phi)$: ω_0 (rad/sec)
   $x[n] = A\cos(\omega_0 n + \phi)$: ω_0 (rad) or (rad/sample)

   (b) Frequency range:

$$\begin{cases} (i)\ |A|e^{j\{(\omega_0+2\pi r)n+\phi\}} = |A|e^{j\omega_0 n} \cdot e^{j2\pi r n} \cdot e^{j\phi} = |A|e^{j(\omega_0 n+\phi} \\ \\ (ii)\ A\cos\{(\omega_0 + 2\pi r)n + \phi\} = A\cos(\omega_0 n + 2\pi r n + \phi) = A\cos(\omega_0 n + \phi) \end{cases}$$

   NOTE:

   The frequency $\omega_0 + 2\pi r$ is indistinguishable from frequency ω_0, i.e. they are the SAME frequencies!!!

   $\Rightarrow$ The only frequency interval that we have to consider for (exponential & sinusoidal) discrete signals is of length 2π, $\ni$: $0 \le \omega_0 \le 2\pi$. [3] [4]

Figure 2.5: Frequency range for discrete-time signals.

[2] The differences are due to the fact that n is an integer, i.e. dimensionless.
[3] Since dim(ω_0)=radian, ω_0 should be between 0 and 2π.
[4] For continuous signals, $A\cos(\omega_0 t + \phi)$, ω_0 could be $0 \le \omega_0 < \infty$.

(c) Peridicity:

Definition 2.1 *A sequence $x[n]$ is called periodic if*

$$x[n] = x[n+N] \ \ \forall n \quad \textit{where } N : \textit{integer}$$

The continuous complex exponential signals and the sinusoids are *always periodic*, and the corresponding period is $T = \frac{2\pi}{\omega_0}$, i.e. [5]

$$e^{j\omega_0 t} = \cos(\omega_0 t) + \sin(\omega_0 t) : \quad \text{periodic} \ \ (T = \frac{2\pi}{\omega_0}(\text{sec}))$$

However, for discrete complex exponentials and sinusoids to be periodic, we have:

$$\begin{cases} (i) \ e^{j\omega_0 n} = e^{j\omega_0 (n+N)} = e^{j\omega_0 n} \cdot e^{j\omega_0 N} \\ \\ (ii) \ \cos(\omega_0 n + \phi) = \cos\{\omega_0 (n+N) + \phi\} = \cos(\omega_0 n + \omega_0 N + \phi) \end{cases}$$

For (i) and (ii) to be valid, the necessary condition is as follows:

$$\omega_0 N = 2\pi k \quad \Rightarrow \quad N = \frac{2\pi}{\omega_0} k \ \text{(should be an integer!!!)} \tag{2.1}$$

Therefore:

(1) The period N may not necessarily be $\frac{2\pi}{\omega_0}$, since $\frac{2\pi}{\omega_0}$ may not be an integer.

(2) May not be periodic at all depending on the frequency ω_0.

[5] $e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$, and therefore $\omega_0 T = 2\pi n \ \rightarrow \ T = \frac{2\pi n}{\omega_0}$. We call $T = \frac{2\pi}{\omega_0}$ the *fundamental period* of the signal.

Example 2.1 *(periodicity of discrete sinusoids)*

(1) $x_1[n] = \cos(\frac{3}{4}\pi n) = \cos(t)|_{t=\frac{3\pi}{4}n}$ $\quad T = 6\pi(\text{sec})$

(2) $x_2[n] = \cos(n) = \cos(t)|_{t=n}$ $\quad$ Non-periodic

Solution:

(1) $\omega_0 = \frac{3\pi}{4}$:

$\frac{3\pi}{4}N = 2\pi k \quad \rightarrow \quad$ smallest integer value $N = 8(\neq \frac{4}{3\pi}2\pi = \frac{8}{3})$

Figure 2.6: $x_1[n] = \cos(\frac{3}{4}\pi n)$

(2) $\omega_0 = 1$:

$N = 2\pi k \quad \rightarrow \quad \exists$ no such integer value $N = 8$

Figure 2.7: $x_2[n] = \cos(n)$

Remarks:

1. Combining (b) and (c) along with (2.1), we can see that for periodic(N: which is given or fixed) complex exponential or real sinusoidal sequences,

   $\exists$ **only N possible frequency components** [6] [7] $\ni$:

$$\begin{aligned}\omega_k &= \frac{2\pi}{N}k \quad \text{where } k = 0, 1, 2, \ldots, N-1 \\ &= k \cdot \omega_f\end{aligned}$$

   (cf) Note that $\omega_k = \omega_{k+N} = 2\pi + \omega_k$, since ω_k and $2\pi + \omega_k$ are the same frequencies, and we call ω_f the *fundamental frequency.*

[6] As we will see later, this is the reason why the discrete Fourier series(DFS) is a finite series.

[7] Continuous Fourier series(CFS) is an infinite series.

2. Since $\omega_0 = \omega_0 + 2\pi k$, for complex exponential and real sinusoidal sequences, frequency around ω_{DC}, where

$$\omega_{DC} = 2\pi k \quad k = \text{integer}$$

represent the *low* frequencies increasing up to $\omega_H = 2\pi k + \pi$, which is the *highest* frequency possible.

(cf): The frequency of continuous sinusoids $A\cos(\Omega_0 t + \phi)$ could be arbitrarily large as Ω_0 increases.

Example 2.2

$$x[n] = \cos(\omega_0 n)$$

(a) $\omega_0 = 0,\quad x[n] = 1\ \forall n$

(b) $\omega_0 = \frac{\pi}{4},\quad x[n] = \cos(\frac{\pi}{4}n)$ (oscillation becomes more rapid)

(c) $\omega_0 = \pi,\quad x[n] = \cos(\pi n)$ (highest frequency possible)

(d) $\omega_0 = \frac{7\pi}{4},\quad x[n] = \cos(\frac{7\pi}{4}n) = \cos(\frac{\pi}{4}n)$ $(\omega_0 = \frac{7\pi}{4} = \frac{\pi}{4})$ [8]

(e) $\omega_0 = 2\pi,\quad x[n] = 1\ \forall n,$ $(\omega_0 = 2\pi = 0)$

[8] $\cos(\frac{7\pi}{4}n) = \cos(2\pi n - \frac{\pi}{4}n) = \cos(\frac{\pi}{4}n)$

2.3 Discrete-time systems

A discrete-time system is generally represented by the following block diagram, where the input sequence is $x[n]$ and the output sequence is $y[n]$:

$$y[n] = T\{x[n]\}$$

Figure 2.8: Block diagram of a discrete-time system

Typical classes of systems

1. **Memoryless system(Static system):**

   A system is called "memoryless" if n-th sample of the output $y[n]$ depends only on the n-th sample of the input $x[n]$.

Example 2.3

(1) $y[n] = T\{x[n]\} = x^2[n]$:square device(memotyless)

(2) $y[n] = T\{x[n]\} = x[n - n_0]$:ideal delay(system w/ memory)

2. **Linear system:**

A system is "linear" if the following condition is satisfied:

$$
\begin{aligned}
T\{\sum_{k=1}^{M} a_k x_k[n]\} &= \sum_{k=1}^{M} a_k T\{x_k[n]\} \\
&= \sum_{k=1}^{M} a_k y_k[n]
\end{aligned}
$$

Example 2.4

Consider a mixer, or juicer which we use in our everyday life:

(1) You put a fresh apple, a tomato, and pear altogether in the mixer and make a cup of fruit juice.

(2) You put a fresh apple in a mixer and make an apple juice, then you make a fresh tomato juice, and pear juice. After that you mix them and make a cup of fruit juice.

Would those two cups of juices taste same? Could we then consider the system(mixer) as a linear system?

3. **Time invariant system:**(Shift invariant system)

A system is called "time-invariant" if:

$$T\{x[n-n_0]\} = y[n-n_0] \quad \forall n_0 : \text{integer}$$

where $y[n] = T\{x[n]\}$.

Example 2.5

Consider a mixer, or juicer which we use in our everyday life:

(1) You keep a fresh apple for a week, and make a cup of apple juice.

(2) You make a fresh cup of apple juice and keep it for a week.

Would those two cups of juices taste same? Could we then consider the system(mixer) as a time-invariant system?

Example 2.6 *Compressor* [9] [10]

$$y[n] = x[Mn] \quad \textit{where } M \textit{ is an integer}$$

Figure 2.9: A discrete-time system: compressor

Determine whether the above system(compressor) is shift-invariant or not.

Solution:

$$\begin{aligned} T\{x[n-n_0]\} &= x[Mn-n_0] \\ &\neq x[Mn-Mn_0] \\ &= y[n-n_0] \end{aligned}$$

Therefore, the system is NOT shift-invariant (unless $M = 1$).

Example 2.7

Determine whether the following system is shift-invariant or not:

$$y[n] = T\{x[n]\}x[n^2]$$

Solution:

$$\begin{aligned} T\{x[n-n_0]\} &= x[n^2-n_0] \\ &\neq x[(n-n_0)^2] \\ &= y[n-n_0] \end{aligned}$$

Therefore, the system is NOT shift-invariant.

[9] This is related to the *time scaling* property of Fourier transform we mentioned in Signals & Systems class. The output $y[n]$ is merely a sequence of resampling every $M-th$ points in $x[n]$, and the system is called a "compressor" although the term is not exact in a rigorous sense.
We should interprete the term "compressor" meaning reducing the amount of data to represent a sequence.

[10] Ina compressor, the time index n is replace by Mn.

4. **Causal system:**(non-anticipative)

   A system is called "causal" if:

$$y[n_0] = f\{x[n]\} \quad \text{where } n \le n_0$$

Example 2.8

$$\begin{array}{ll} (1)\ y[n] = T\{x[n]\} = x[n] - x[n-1] & \text{:backward difference system(c)} \\ (2)\ y[n] = T\{x[n]\} = x[n+1] - x[n] & \text{:forward difference system(n-c)} \\ (3)\ y[n] = T\{x[n]\} = x[Mn] & \text{:compressor(n-c, unless } M = 1) \\ (4) & \end{array}$$

$$y[n] = T\{x[n]\} = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

:moving average(n-c, unless $M_1 = 0$) [11]

5. **Stable system:**

   A system is called "stable" if and only if (iff) a bounded input produces a bounded output sequence.

   Note: A sequence $x[n]$ is bounded if $\exists$ a positive finite value B_x $\ni$: $|x[n]| \le B_x < \infty\ \forall n$.

Example 2.9 *Accumulator* [12]

Determine the system defined below is a stable system or not.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Solution: Suppose $x[n] = u[n]$, then the input $x[n]$ is bounded since $\exists\ B_x$ (*e.g.* $B_x = 2$) $\ni$: $|x[n]| \le B_x < \infty\ \forall n$.
The corresponding output sequence of the system is given as:

$$y[n] = \begin{cases} 0 & n < 0 \\ n+1 & n \ge 0 \end{cases}$$

Since there does NOT $\exists\ B_y$ $\ni$: $n = 1 \le B_y\ \forall n$, $y[n]$ is not bounded. Therefore, the system is NOT a stable system.

[11] Average of present sample, past M_2 samples, and future M_1 samples.

[12] Recall that $u[n] = \sum_{-\infty}^{n} \delta[k]$.

2.4 LTI(Linear Time Invariant) system

A system $T\{\cdot\}$ is called an LTI system if it satisfies both the *linearity* and *time-invariant* properties:

1. Linearity:
$$T\{\sum_{k=1}^{M} a_k x_k[n]\} = \sum_{k=1}^{M} a_k T\{x_k[n]\}$$

2. Time Invariance:
$$T\{x[n-n_d]\}y[n-n_d] \quad \text{where } y[n] = T\{x[n]\}$$

Remarks:

(1) An LTI system $T\{\cdot\}$ is completely specified(characterized) by its impulse response $h[n]$ where $h[n] \triangleq T\{\delta[n]\}$

(2) The output sequence $y[n]$ od an LTI system can be represented as a "convolution sum" b/w input $x[n]$ and the system's impulse response $h[n]$, i.e.:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \triangleq x[n] * h[n]$$

Derivation:

Figure 2.10: An LTI system

Recall that any sequence $x[n]$ can be expressed as a linear combination of weighted and shifted impulses, i.e.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Therefore, the output sequence $y[n]$ of an LYI system for an arbitrary input sequence $x[n]$ is as follows:

$$\begin{aligned}
y[n] = T\{x[n]\} &= T\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \quad \text{(linearity)} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{(time-invariance)} \\
&\triangleq x[n] * h[n]
\end{aligned}$$

:Convolution Sum

Interpretation of convolution sum:

Figure 2.11: DLTI system

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{2.2}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.3}$$

From (2.2) and (2.3), we can see that each sample of the input sequence (i.e. $x_k[n] = x[k]\delta[n-k]$) is *transformed* into output sample $y_k[n] = x[k]h[n-k]$, and the results are summed up to form the output sequence $y[n]$.

Example 2.10 Consider a DLTI system and find out the output sequence $y[n]$, where the input $x[n]$ and the inpulse response $h[n]$ of the system are respectively given as follows: $x[n] = \delta[n+1] - \delta[n-1] \triangleq x_{-1}[n] + x_1[n]$
$h[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

Figure 2.12: Input $x[n]$ and impulse response $h[n]$ of a DLTI system.

Solution: According to the above interpretation of convolution sum, we have:

$$y[n] = x[-1]h[n+1] + x[1]h[n-1] \triangleq y_{-1}[n] + y_1[n]$$

which is depicted in the following figure.

Figure 2.13: Output $y[n]$.

Evaluation of convolution sum: all in k domain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

(1) Flip $h[k]$ around $k = 0$ to obtain $h[-k]$.

(2) Shift $h[-k]$ in amount of n to obtain $h[n-k]$.

(3) Multiply $x[k]$ to $h[n-k]$, and take sum of them to obtain $y[n]$ for a fixed(or certain range of) n.

Example 2.11 Repeat the previous example 2.10, and find the output sequence by directly evaluating the convolution sum.

Solution: The output expressed in the form of convolution sum is as follows:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

We must calculate this summation for six different cases of n:

Figure 2.14: Convolution sum in k domain.

(a) $n \leq -2$; $y[n] = 0$

(b) $n = -1$; $y[-1] = 1 \times 1 = 1$

(c) $n = 0$; $y[0] = 1 \times \frac{1}{2} = \frac{1}{2}$

(d) $n = 1$; $y[1] = -1 \times 1 = -1$

(e) $n = 2$; $y[2] = -1 \times \frac{1}{2} = -\frac{1}{2}$

(f) $n \geq 3$; $y[n] = 0$

Figure 2.15: Output $y[n]$ via convolution sum.

2.5 Properties of LTI system

where $h[n] = T\{\delta[n]\}$

Figure 2.16: A typical LTI system

Recall that an LTI system is completely characterized by its impulse response $h[n]$. Therefore:

$$\textbf{properties of LTI system} \equiv \textbf{properties of h}[\mathbf{n}]$$
$$\textbf{(w/ properties of the convolution sum)}$$

1. **Commutative property:**

$$h[n] * x[n] = x[n] * h[n]$$

   (a) $\sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.
   (b) The prrof is trivial. (assignment)
   (c) Roles of $x[n]$ and $h[n]$ can be interchanged, i.e. two different systems may have the same output, depending on the input. ($y[n] = x[n] * h[n] = h[n] * x[n]$)

2. **Distributive property:**

$$x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n]$$

   proof: trivial (assignment)

Remarks:

(a) Cascade of LTI systems:

$$\text{where } h[n] = h_1[n] * h_2[n] * \ldots * h_N[n]$$

Figure 2.17: Cascade of LTI systems.

(i) proof is trivial (assignment)
(ii) The order of subsystems is irrelevant(due to commutativity).

(b) Parallel connection of LTI systems:

$$\text{where } h[n] = \sum_{i=1}^{N} h_i[n]$$

Figure 2.18: Parallel connection of LTI systems(e.g. filter banks).

(i) proof is trivial (assignment)
(ii) The result is due to distributivity of LTI system.

3. **Stable LTI system:**

An LTI system is stable iff $h[n]$ is absolutely summable. i.e.

$$\sum_{k=-\infty}^{\infty} |h[k]| = S < \infty$$

Note: The stability condition of LTI system is in terms of $h[n]$ only!!!

Proof:

(a) If ($\leftarrow$):

Suppose $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, and let the input $x[n]$ is bounded, i.e. $|x[n]| \leq B_x$, then we must prove that the output $y[n]$ is also bounded.

$$\begin{aligned} |y[n]| = |\sum_{k=-\infty}^{\infty} h[k]x[n-k]| &\leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \\ &\leq \sum_{k=-\infty}^{\infty} |h[k]| \cdot B_x \\ &< \infty \ \ (\text{since} \sum |h[k]| < \infty,\ B_x < \infty) \end{aligned}$$

Therefore, $y[n]$ is bounded, and the system is stable.

(b) Only if ($\rightarrow$):

This is equivalent to proving; if $\sum_{k=-\infty}^{\infty} |h[k]| \rightarrow \infty$, the system is unstable, i.e. bounded $x[n]$ may cause unbounded $y[n]$. [13]

Take an input sequence $x[n]$ as:

$$x[n] \triangleq \begin{cases} \frac{h^*[-n]}{|h[-n]|} & \text{if } h[n] \neq 0 \\ 0 & \text{if } h[n] = 0 \end{cases}$$

Clearly, $|x[n]| \leq 1 = B_x < \infty$ (bounded), but

$$y[0] = \sum_{k=-\infty}^{\infty} x[0-k]h[k] = \sum_{k=-\infty}^{\infty} \frac{h^*[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| \rightarrow \infty$$

Therefore, the system is unstable. (Q.E.D.)

[13] Antithesis: $A \rightarrow B \ \equiv \ \overline{B} \rightarrow \overline{A}$.

4. **Causal LTI sysrem:**

   An LTI system is causal if

$$h[n] = 0 \quad \forall n < 0$$

Note: Again, notice that the causality condition of LTI system is in terms of $h[n]$ only!!!

Proof:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$\begin{aligned} \rightarrow \quad y[n_0] &= \sum_{k=-\infty}^{\infty} h[k]x[n_0-k] \\ &= \sum_{k=0}^{\infty} h[k]x[n_0-k] \quad \text{(by condition)} \end{aligned}$$

Since $y[n_0]$ depends only on the input $x[n]$ for $n \le n_0$, the system is causal.

Example 2.12 Find out whether the following systems are stable, and causal. (Be reminded that the impulse response of the system is defined as: $h[n] \triangleq T\{\delta[n]\}$.)

(i) Ideal Delay: $y[n] = x[n-n_d]$, where $n_d > 0$ (integer)
(ii) Accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$
(iii) Forward difference: $y[n] = x[n+1] - x[n]$
(iv) Backward difference: $y[n] = x[n] - x[n-1]$

Solution:

(i) Ideal Delay: $y[n] = x[n-n_d]$, where $n_d > 0$ (integer)

Applying the definition of the impulse response, we have;

$$h[n] = T\{\delta[n]\} = \delta[n-n_d]$$

(a) $\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} \delta[n-n_d] = 1 < \infty$:stable.
(b) $h[n] = 0 \ \forall\, n < 0$:causal

Remark: The system's impulse response $h[n]$ has a finite number of non-zero samples(i.e. finite duration): called a FIR (finite impulse response) system.
$\Rightarrow$ An FIR system is always stable if each sample in $h[n]$ is of finite magnitude.

(ii) Accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$

Using the definition of the impulse response, we have;

$$h[n] = T\{\delta[n]\} = \sum_{k=-\infty}^{n} \delta[k] = \left\{ \begin{array}{ll} 1 & n \geq 0 \\ 0 & n < 0 \end{array} \right\} = u[n]$$

(a) $\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} u[k] = \sum_{k=0}^{\infty} u[k] \rightarrow \infty$:unstable.
(b) $h[n] = u[n] = 0 \;\; \forall\, n < 0$:causal

Remark: $h[n]$ has infinite number of non-zero samples: called an IIR (infinite impulse response) system.
$\Rightarrow$ could be a stable system. (i.e. not always unstable even though it has ∞ number of samples.)

(e.g.)

$$h[n] = a^n u[n], \quad |a| < 1$$

$$\begin{aligned} \rightarrow \sum_{k=-\infty}^{\infty} |h[k]| &= \sum_{k=0}^{\infty} |a|^k \\ &= \frac{1}{1-|a|} < \infty \end{aligned}$$

Figure 2.19: Impulse responses of a stable and an unstable LTI systems.

(iii) Forward difference: $y[n] = x[n+1] - x[n]$

By the definition of the impulse response, we have;

$$h[n] = T\{\delta[n]\} = \delta[n+1] - \delta[n]$$

(a) $\sum_{k=-\infty}^{\infty} |h[k]| = 2 < \infty$:stable.
(b) $h[n] = 1 \neq 0$ for $n = -1$:non-causal

(iv) Backward difference: $y[n] = x[n] - x[n-1]$

By the definition of the impulse response, we have;

$$h[n] = T\{\delta[n]\} = \delta[n] - \delta[n-1]$$

(a) $\sum_{k=-\infty}^{\infty} |h[k]| = 2 < \infty$:stable.

(b) $h[n] = 0 \neq 0 \ \forall\, n < 0$:causal

5. **Inverse sysrem:**

An LTI system with impulse response $h_i[n]$ is called the inverse system od another LTI system w/ $h[n]$ if:

$$h_i[n] * h[n] = h[n] * h_i[n] = \delta[n] \tag{2.4}$$

Example 2.13

Figure 2.20: Cascade of the accumulator($h_1[n]$) and the backward difference($h_2[n]$) systems.

$$\begin{aligned} h_1[n] * h_2[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\ &= u[n] - u[n-1] \\ &= \delta[n] \end{aligned}$$

Therefore, the accumulator and the backward difference systems are inverse systems to each other. [14]

Remark: Solving (2.4) directly in time to find an inverse system $h_i[n]$ for a $h[n]$ is difficult in general.

⇒ Solving in frequency domain using Z-transform will make the job easier for you! (will be covered later at Chapter 5.)

[14] This result make sense that the accunulator stacks up the incoming samples while the backward difference system sequentially pulls out the samples one at a time, which means that the system does not have any effect at all on the input signal.

2.6 Linear constant coefficient difference equations

For some LTI systems, the input $x[n]$ and the output $y[n]$ are related in an N-th order linear constant coefficient difference equation, i.e.: [15]

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{2.5}$$

Representation of an LTI system in (2.5):

Example 2.14

Accumulator:

From the previous example, we know that the accumulator and the backward difference systems are inverse systems to each other, i.e.:

Figure 2.21: The accumulator($h_1[n]$) and the backward difference($h_2[n]$) systems in cascade.

Therefore, from the input/output relation of the backward difference system in the above figure, we have

$$y[n] - y[n-1] = x[n], \quad : \text{I/O relation of accumulator in the form of (2.5)}$$

where $a_0 = 1, a_1 = -1, M = 0, N = 1, b_0 = 1$.

Or, from the i/o relation of the accumulator; [16]

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} x[k] \\ &= x[n] + \sum_{k=-\infty}^{n-1} x[k] \\ &= x[n] - y[n-1] \end{aligned}$$

$$\Rightarrow y[n] - y[n-1] = x[n]$$

[15] This will be useful when we discuss the structures of digital filters later.

[16] This is the method in your textbook.

Rearranging the above I/O realtion of the accumulator, we get [17] [18]

$$y[n] = y[n-1] + x[n] \quad \text{(Recursion representation)} \tag{2.6}$$

Figure 2.22: Accumulator in recursive form.

Example 2.15

Moving average:

Recall that the I/O relation of the moving average system is as follows:

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

Consider that case when $M_1 - = 0$ to make the system be causal, i.e.

$$y[n] = \frac{1}{M_2 + 1} \sum_{k=0}^{M_2} x[n-k] \tag{2.7}$$

Notice that (2.7) itself is in the form of (2.5), where $N = 0, a_0 = 1, b_k = \frac{1}{M_2+1}$ for $0 \leq k \leq M_2$.

Another possibility: [19]

The impulse response of the system is now

$$\begin{aligned} h[n] = \frac{1}{M_2+1} \sum_{k=0}^{M_2} \delta[n-k] &= \begin{cases} 0, & n < 0 \\ \frac{1}{M_2+1}, & M_2 \geq n \geq 0 \\ 0, & n > M_2 \end{cases} \\ &= \frac{1}{M_2+1} \{u[n] - u[n - M_2 - 1]\} \\ &= \frac{1}{M_2+1} \{\delta[n] - \delta[n - M_2 - 1]\} * u[n] \end{aligned}$$

[17] This form clearly reveals the reason why the system is called accumulator.

[18] Representing the system's i/o in (2.5) is NOT unique. In fact we replace $y[n-1] = y[n-2] + x[n-1]$ in (2.6), we get another form of recursion representation as: $y[n-1] - y[n-2] = x[n] + x[n-1]$, which mean there are infinitely many ways of representation!!!

[19] Representation of an LTI system in (2.5) is not unique, and in fact there exist infinitely many ways, and details including the solution of (2.5) will be discussed later at Chapter 6.

Figure 2.23: Impulse response $h[n]$ of a causal moving average system.

Therefore, the system acn be represented in the following block diagram which is in a cascade form:

Figure 2.24: A causal moving average system in cascade of two subsystems.

$$y[n] - y[n-1] = x_1[n] \quad \text{(from previous example)} \tag{2.8}$$

$$x_1[n] = \frac{1}{M_2+1}\{x[n] - x[n-M_2-1]\} \tag{2.9}$$

Inserting (2.9) into (2.8), we get

$$y[n] - y[n-1] = \frac{1}{M_2+1}\{x[n] - x[n-M_2-1]\} \tag{2.10}$$

(2.10) is also in the form of (2.5) where $a_0 = 1, a_1 = -1, N = 1, b_0 = -b_{M_2+1} = \frac{1}{M_2+1}$.

2.7 Frequency domain representation of discrete-time signals and systems

Consider an LTI system w/ an input $x[n] = e^{j\omega n}$, i.e. complex exponential sequence of frequency ω(rad). [20]

Figure 2.25: LTI system w/ input $x[n] = e^{j\omega n}$.

$$
\begin{aligned}
y[n] = h[n] * x[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} \\
&= e^{j\omega n} \cdot \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \right) \\
&= e^{j\omega n} \cdot H(e^{j\omega}) \\
&= x[n] \cdot H(e^{j\omega})
\end{aligned}
$$

where

$$H(e^{j\omega}) \triangleq \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

In summary, we have

$$T\{x[n]\} = H(e^{j\omega}) \cdot x[n]$$

$$\Rightarrow T\{e^{j\omega n}\} = H(e^{j\omega}) \cdot e^{j\omega n}$$

Note that the output of the LTI system w/ sinusoidal input is again sinusoidal with the same frequency. So, representation of signals associated with LTI systems using sinusoids is very important!!!

[20] $x[n] = e^{j\omega n} = \cos(\omega n) + j\sin(\omega n)$.

Remarks:

1. $H(e^{j\omega})$ is the DTFT of $h[n]$. (Recall from Signals and Systems class...)
2. $e^{j\omega n}$ is the *eigenfunction* and $H(e^{j\omega})$ is the *eigenvalue* of the LTI system. [21]
3. $H(e^{j\omega})$ represents the change in complex magnitude of $e^{j\omega n}$ as a function of frequency ω, and called the **frequency response** of the system:

$$\begin{aligned} H(e^{j\omega}) &= H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\ &= |H(e^{j\omega})|e^{j\Phi_H(e^{j\omega})} \end{aligned}$$

where

$$\Phi_H(e^{j\omega}) = \arctan \frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}$$

Example 2.16

Determine the frequency response of the ideal delay system.

Figure 2.26: An ideal delay system of n_d samples.

Solution:

$$y[n] = x[n - n_d]$$

(#1) Let $x[n] = e^{j\omega n}$, then

$$\begin{aligned} y[n] = e^{j\omega(n-n_d)} &= e^{j\omega n} \cdot e^{-j\omega n_d} \\ &= e^{j\omega n} \cdot H(e^{j\omega}) \end{aligned}$$

Therefore, we have;

$$\begin{aligned} H(e^{j\omega}) = e^{-j\omega n_d} &= \cos(\omega n_d) - j\sin(\omega n_d) \equiv H_R(e^{j\omega}) + jH_I(e^{j\omega}) \\ &= 1 \cdot e^{j(-\omega n_d)} \qquad\qquad \equiv |H(e^{j\omega})|e^{j\Phi_H(e^{j\omega})} \end{aligned}$$

[21] In linear algebra, for a square matrix A, if it satisfies $y = Ax = \lambda x$, x and λ are called the *eigenvector* and the *eigenvalue* of the matrix A. In this case, A is the system, x is the input, and $y = Ax$ corresponds to the output of an LTI system as an anology.

(#2) Since $h[n] = \delta[n - n_d]$, we get

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n - n_d]e^{-j\omega n} = e^{-j\omega n_d}$$

Remark:
The concept of frequency response for a discrete system is very similar to the transfer function of a continuous system, i.e.

$$H(\Omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt$$

But, the main difference is $H(e^{j\omega})$ is periodic in ω with period of 2π, i.e. [22]

$$H(e^{j\omega}) = H(e^{j(\omega+2\pi k)}), \quad \forall\ k\text{:integer}$$

Proof: Assignment(easy and already done at S&S class.)

(cf)

1. The fact is due to that $e^{j\omega n}$ is periodic in ω (2π), so the system $H(e^{j\omega})$ should respond identically to $e^{j\omega n}$ and $e^{j(\omega+2\pi k)n}$.
2. Usually, we specify $H(e^{j\omega})$ for $-\pi \leq \omega \leq \pi$.

[22] As a matter of fact, DTFT of any sequence is periodic(2π).

Example 2.17

Frequency response ideal filters:

(a) Ideal LPF:

(b) Ideal BPF:

(c) Ideal HPF:

(d) Ideal BSF(band stop filter):

Figure 2.27: Ideal LPF, BPF, HPF, and BSP.

2.8 Representation of sequences by Fourier transforms(Discrete time Fourier transform: DTFT)

Recall that the DTFT pair for a non-periodic sequence $x[n]$ is defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{:analysis} \tag{2.11}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{:synthesis} \tag{2.12}$$

$$\Big(x[n] \quad \overset{F}{\longleftrightarrow} \quad X(e^{j\omega})\Big)$$

In general, there are two ways of representing the DTFT:

$$\begin{aligned} X(e^{j\omega}) &= X_R(e^{j\omega}) + X_I(e^{j\omega}) && \text{:rectangular form} \\ &\overset{\text{or}}{=} |X(e^{j\omega})|e^{j\Phi_X(e^{j\omega})} && \text{:polar form} \end{aligned}$$

where $|X(e^{j\omega})|$ and $\Phi_X(e^{j\omega})$ are called the *magnitude spectrum* and the *phase spectrum* respectively.

Remark:

1. The phase $\Phi_X(e^{j\omega})$ is not unique, since the phase angle rotates for every 2π(rad), i.e.

$$e^{j\Phi_X(e^{j\omega})} = e^{j[\Phi_X(e^{j\omega})+2\pi k]}$$

2. Also recall again that $X(e^{j\omega})$ is periodic in ω w/ period of 2π, i.e.

$$\Phi_X(e^{j\omega}) = \Phi_X(e^{j(\omega+2\pi k)})$$

3. Representation:

   (i) The principal value(within $[-\pi, \pi]$) of the phase:

$$\Phi_X(e^{j\omega}) = \text{ARG}\Big[X(e^{j\omega})\Big] \quad \text{for} \quad -\pi \leq \Phi_X(e^{j\omega}) \leq \pi$$

   (ii) The phase value in the main period(since it is periodic):

$$\Phi_X(e^{j\omega}) = \arg\Big[X(e^{j\omega})\Big] \quad \text{for} \;\; 0 \leq \omega \leq \pi$$

Derivation of (2.11) and (2.12): Done at S&S class

Verification of (2.11) and (2.12):

Let $\hat{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\right)e^{j\omega m}d\omega$, then we want to show that :

$$\hat{x}[n] = x[n]$$

proof:[23]

$$\begin{aligned}
\text{LHS} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m}\cdot e^{j\omega m}d\omega \\
&= \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega\right] \\
&= \sum_{m=-\infty}^{\infty} x[m]\frac{\sin[\pi(n-m)]}{\pi(n-m)} \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \\
&= x[n] \\
&= \text{RHS}
\end{aligned}$$

Condition of $x[n]$ to have $X(e^{j\omega})$: sufficient condition

For a sequence $x[n]$ to be represented by (2.12) (IDTFT: i.e. for $X(e^{j\omega})$ to exists), the infinite sun in (2.11) should converge, i.e. $|X(e^{j\omega})| < \infty$. Therefore, we have:

$$\begin{aligned}
|X(e^{j\omega})| &= |\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}| \\
&\leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| \\
&= \sum_{n=-\infty}^{\infty} |x[n]|
\end{aligned}$$

$\Rightarrow$ If $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$, then $|X(e^{j\omega})| < \infty$.

$\Rightarrow$ If $x[n]$ is "absolutely summable" (i.e. $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$), then $X(e^{j\omega})$ exists!!!!!

[23]Note that $\frac{\sin[\pi(n-m)]}{\pi(n-m)} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$

(cf)

1. If a sequence $x[n]$ is a finite duration/magnitude sequence, $X(e^{j\omega})$ exists.
2. If a DLTI systems $h[n]$ is stable, the frequency response $H(e^{j\omega})$ exists.(e.g. FIR system)

Example 2.18

Find the DTFT of $x[n] = a^n u[n]$.

Solution:

$$\begin{aligned} X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} &= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n \\ &= \frac{1}{1 - ae^{-j\omega}} \quad \text{if } |ae^{-j\omega}| < 1 \ \ (\text{or } \ |a| < 1) \end{aligned}$$

(cf) Absolute summability on $x[n]$:

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1 - |a|} \quad \text{if } |a| < 1$$

Note that the above two conditions are equivalent, and we can conclude that if $x[n]$ is absolutely summable, then $X(e^{j\omega})$ exists.

Singular sequences:

Sequences which are not absolutely summable, but still have their own DTFT.

1. $x[n] = 1 \quad \forall \ n$:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

where $\delta(\omega)$ is the continuous unit impulse function of which definition is as follows:

$$\delta(\omega) = \begin{cases} \infty, & \omega = 0 \\ 0, & \omega \neq 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(\omega) d\omega = 1$$

verification:

Let $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$, where $|\omega_0| \leq \pi$, then

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega - \omega_0)e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} 2\pi e^{j\omega_0 n} \\ &= e^{j\omega_0 n} \end{aligned}$$

$$\left(\text{i.e. } e^{j\omega_0 n} \stackrel{F}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k) \right)$$

By letting $\omega_0 = 0$, the relationship is proved.

Figure 2.28: DTFT of discrete d-c signal.

2. $x[n] = \sum_m a_m e^{j\omega_m n} \quad -\infty < n < \infty$:

   From the above verification, it is obvious that: [24]

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi \sum_m a_m \delta(\omega - \omega_m + 2\pi k)$$

3. $x[n] = u[n]$: unit step sequence [25] [26]

$$X(e^{j\omega}) = U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega + 2\pi r)$$

derivation: assignment

[24]Due to the linearity property of DTFT to be discussed later.

[25]Note that $u[n]$ is not absolutely summable, and $\sum_{n=-\infty}^{\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \left(e^{-j\omega}\right)^n \rightarrow \infty$, since $|e^{-j\omega}| = 1$.

[26]Notice the similarity b/w the DTFT of $u[n]$ and the FT of the unit step function $u(t)$, which is $\mathcal{F}[u(t)] = U(\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega}$.

2.9 Symmetry properties of DTFT

Definition 2.2 A complex sequence $x_e[n]$(or a complex function $X_e(\omega)$) is called *conjugate symmetric* if:

$$x_e[n] = x_e^*[-n] \quad \text{or} \quad X_e(\omega) = X_e^*(-\omega)$$

(cf)

1. In other words, the real part is even, whereas the imaginary part is odd.
2. If the sequence and/or the function is *real*, called *even* sequence(or function).

Definition 2.3 A complex sequence $x_o[n]$(or a complex function $X_o(\omega)$) is called *conjugate anti-symmetric* if:

$$x_o[n] = -x_o^*[-n] \quad \text{or} \quad X_o(\omega) = -X_o^*(-\omega)$$

(cf)

1. In other words, the real part is odd, whereas the imaginary part is even.
2. If the sequence and/or the function is *real*, called *odd* sequence(or function).

FACT:
Any sequence $x[n]$ and its DTFT $X(e^{j\omega})$ can be decomposed into the sum of conjugate symmetric and the conjugate anti-symmetric parts:

1. $x[n] = x_e[n] + x_o[n]$

$$\begin{aligned} \text{where} \quad x_e[n] &= \frac{1}{2}\{x[n] + x^*[-n]\} \\ x_o[n] &= \frac{1}{2}\{x[n] - x^*[-n]\} \end{aligned}$$

2. $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$

$$\begin{aligned} \text{where} \quad X_e(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{j\omega}) + X^*(e^{j\omega})\right\} \\ X_o(e^{j\omega}) &= \frac{1}{2}\left\{X(e^{j\omega}) - X^*(e^{j\omega})\right\} \end{aligned}$$

proof: assignment

Assignment: Table 2.1 at p.56 needs to be self studied and verified, including the relevant examples.

2.10 Other properties of DTFT: Review

Let [27]

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega})$$

1. Linearity:

$$F\left\{\sum_{k=1}^{M} a_k x_k[n]\right\} = \sum_{k=1}^{M} a_k F\left\{x_k[n]\right\}$$

2. Time-shift and frequency shift:

$$F\left\{x[n-n_0]\right\} = X(e^{j\omega}) \cdot e^{-j\omega n_0}$$

$$F\left\{x[n] \cdot e^{j\omega_0 n}\right\} = X(e^{j(\omega-\omega_0)})$$

3. Time reversal: [28]

$$F\left\{x[-n]\right\} = X(e^{-j\omega}) \overset{\dagger}{=} X^*(e^{j\omega})$$

4. Differentiation in frequency:

$$F\left\{n \cdot x[n]\right\} = j\frac{d}{d\omega}X(e^{j\omega})$$

5. **Parseval's theorem:**

   The energy E of a sequence $x[n]$ calculated in time and frequency domain are equal:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

   where $|X(e^{j\omega})|^2$ is called the "energy density spectrum" of $x[n]$.

---

[27] Observe the close similarity with those of Fourier transform.

[28] † The second equality is if $x[n]$ is real. For real $x[n]$, $X(e^{j\omega})$ is conjugate symmetric, i.e. $\mathrm{Re}[X(e^{j\omega})]$ is even, and $\mathrm{Im}[X(e^{j\omega})]$ is odd; refer the Table 2.1.

6. Convolution:

$$
\begin{aligned}
F\{x[n]w[n]\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})W(e^{j(\omega-\Omega)})d\Omega \\
&= \frac{1}{2\pi} X(e^{j\omega}) * W(e^{j\omega}) \qquad \text{:periodic convolution}
\end{aligned}
$$

proof: assignments

Self study:

(1) DTFT properties at Table 2.2 in page 59 of your textbook.

(2) Typical DTFT pairs at Table 2.3 in page 62 of the textbook.

(3) Examples from page 63 to page 64 at the textbook.

2.11 Discrete-time random signals

We need quite a lot of backgrounds on probability, random variables, and random processes to deal with discrete-time random signals. We, therefore, omit this section, and you will have opportunities to cover this topic later at the graduate courses, hopefully. [29]

[29] Or, you may study this section for yourself with some references and Appendix A of your textbook.