Contents

Chapter 3

The Z Transform

3.1 Introduction

Z-transform : Generalization of DTFT

Remarks:

1. Certain conditions are needed for DTFT to be defined for discrete signals (e.g. *absolute summability* of $x[n]$).

   $\Rightarrow$ Needs a general transformation for broader class of discrete signals.

2. Laplace transform is a generalization of Fourier transform for continuous signals.

(cf.)

(1) Laplace vs. Fourier transform

(a) $X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

(b) $X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ where $s = \sigma + j\omega$.

Therefore,

$$X(\omega) = X(s)|_{s=j\omega}$$

i.e., $X(\omega)$ corresponds to the $X(s)$ where $\sigma = 0$, which means the sliced version of the Laplace transform along the axis of $\sigma = 0$.

(2) Z transform vs. DTFT

(a) $X(E^{j\omega}) = F\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

(b) $X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ where $z = re^{j\omega}$.

Therefore,

$$X(e^{j\omega}) = X(z)|_{z=ej\omega}$$

i.e., $X(e^{j\omega})$ corresponds to the $X(z)$ where $r = 1$, which means the Z transform along the unit circle on the complex plane of z.

Figure 3.1: Laplace vs. Fourier transform and Z transform vs. DTFT.

3.2 Z transform

Definition 3.1 The z transform $X(z)$ of a discrete-time signal $x[n]$ is defined as follows:

$$\mathcal{Z}\{x[n]\} \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n} \stackrel{d}{=} X(z) \quad \text{: Bilateral z-transform(two sided)}$$

where z is a complex variable, [1] i.e.

$$\begin{aligned} z &= \text{Re}[z] + j\text{Im}[z] \\ &= r \cdot e^{j\omega} \end{aligned}$$

REMARKS:

$$\begin{aligned} X(z) = X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)e^{-j\omega n} \\ &= F\left\{x[n]r^{-n}\right\} \end{aligned}$$

(1) $r = 1 \quad \longrightarrow \quad X(z) = F\{x[n]\}$

Figure 3.2: A unit circle on z plane.

(2) (a) $z = 1 \rightarrow j \rightarrow -1 \rightarrow -j \rightarrow 1$

(b) $\omega = 0 \rightarrow \frac{\pi}{2} \rightarrow \pi \rightarrow \frac{3\pi}{2} \rightarrow 2\pi$

: implies the periodicity of DTFT w/ period 2π.

[1] $\text{Re}[z] = r\cos(\omega)$, and $\text{Im}[z] = r\sin(\omega)$.

(3) Region of concergence(ROC):

$$X(z) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} = F\left\{x[n]r^{-n}\right\}$$

$\Rightarrow$ For the existence of $X(z)$, we need a condition(i.e. *absolute summability* of sequence $x[n]r^{-n}$) as:

$$\sum_{n=-\infty}^{\infty} \left|x[n]r^{-n}\right| < \infty$$

$$\text{OR} \quad \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty \quad \because |z| = r$$

Therefore, we have

$$\begin{aligned} \text{ROC of } X(z) &= \{z|X(z) \text{ exists or converges}\} \\ &= \{z| \sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty\} \\ &: \text{ composed of circles in terms of } |z| \end{aligned}$$

$\Rightarrow$ ROC of $X(z)$ only depends on $|z|$.

$\Rightarrow$ ROC is compose of circles.

$\Rightarrow$ If the unit circle is within the ROC, then DTFT exists.

Figure 3.3: A typical ROC on the z plane.

(4) Most important and useful form of z transform: [2]

$$X(z) = \frac{P(z)}{Q(z)}, \quad \text{where } P(z), Q(z) \text{ are polynomials of } z$$

(a) $\{z|P(z) = 0\}$: *zeros* of $X(z)$: denoted O in z plane.

(b) $\{z|Q(z) = 0\}$: *poles* of $X(z)$: denoted X in z plane.

Example 3.1

Determine the z transform of an exponential sequence given below: [3]

$$x[n] = a^n u[n]$$

Figure 3.4: A right sided exponential sequence $x[n]$ for $0 < a < 1$.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n$$

(a)

$$\text{ROC: } |az^{-1}| < 1 \quad \Rightarrow \quad |z| > |a|$$

(b)

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

[2]For example, in an LTI system where i/o is related in a linear constant coefficient difference equation, $H(z) = \mathcal{Z}\{h[n]\}$ is in the following form:

$$H(z) = \frac{Y(z)}{X(z)}$$

.

[3]Notice that $\exists$ $x[n]$ only for $n \geq 0$, thus a right sided sequence.

Figure 3.5: The ROC of the z transform for a right sided exponential sequence.

Note:

(a) If $|a| < 1 \quad \longrightarrow \quad X(e^{j\omega})$ exists. (Recall !!!)

(b) If $|a| = 1 \quad \overset{\text{e.g. } a=1}{\longrightarrow} \quad x[n] = u[n],$ then

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

(c) poles: $z = a$ (represented by X)
zeros: $z = 0$ (represented by O)

Example 3.2

Determine the z transform of an exponential sequence given below: [4]

$$x[n] = -a^n u[-n-1]$$

Figure 3.6: A left sided exponential sequence $x[n]$ for $a > 1$.

[4]Notice that $\exists\ x[n]$ only for $n < 0$, thus a left sided sequence.

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} \left(a^{-1}z\right)^n$$

(a)

$$\text{ROC: } |a^{-1}z| < 1 \quad \Rightarrow \quad |z| < |a|$$

(b)

$$X(z) = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{z}{z - a}$$

Figure 3.7: The ROC of the z transform for a left sided exponential sequence.

Note:

(a) If $|a| > 1 \quad \longrightarrow \quad X(e^{j\omega})$ exists!

(b) Notice that the z transform $X(z)$ is the same as in the previous example, while the ROC is different.

$\Longrightarrow$ This indicates **the necessity of ROC** for representing the z transform.

(c) poles: $z = a$ (represented by X)

zeros: $z = 0$ (represented by O)

Example 3.3

Determine the z transform of another exponential sequence given below: [5]

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n] \triangleq x_1[n] + x_2[n]$$

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} + \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} \triangleq X_1(z) + X_2(z)$$

$$\text{(a)}\;\; X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \left|\frac{1}{2}z^{-1}\right| < 1 \; (|z| > \frac{1}{2})$$

$$\text{(b)}\;\; X_2(z) = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(-\frac{1}{3}z^{-1}\right)^n = \frac{1}{1+\frac{1}{3}z^{-1}}, \quad \left|\frac{1}{3}z^{-1}\right| < 1 \; (|z| > \frac{1}{3})$$

Therefore, the z-transform $X(z)$ of $x[n]$ is as follows:

$$X(z) = X_1(z) + X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

where

$$\text{ROC} = \left\{z \middle| \left(|z| > \frac{1}{2}\right) \cap \left(|z| > \frac{1}{3}\right)\right\} = \left\{z \middle| |z| > \frac{1}{2}\right\}$$

Figure 3.8: The ROC of the z transform for $x[n]$.

[5]Notice that $\exists\, x[n]$ only for $n \geq 0$, thus a right sided sequence.

Example 3.4

Determine the z transform of the exponential sequence given below: [6]

$$x[n] = \left(-\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1] \triangleq x_1[n] + x_2[n]$$

Solution:

$$\text{(a)} \;\; X_1(z) = \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \frac{1}{3}$$

$$\text{(b)} \;\; X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \frac{1}{2}$$

Therefore, the z-transform $X(z)$ of $x[n]$ is as follows:

$$X(z) = X_1(z) + X_2(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$

where

$$\text{ROC} = \left\{ z \middle| \left(|z| > \frac{1}{3}\right) \cap \left(|z| < \frac{1}{2}\right) \right\} = \left\{ z \middle| \frac{1}{3} < |z| < \frac{1}{2} \right\}$$

Figure 3.9: The ROC of the z transform for two sided $x[n]$.

Note:
From the above examples, notice that if $x[n]$ is a (sum of) infinitely long exponential sequences, then the z-transform $X(z)$ is **a rational function of z^{-1} or z.**

[6] Notice that $\exists\, x[n]$ for entire n, i.e. $-\infty < n < \infty$, thus a **two** sided sequence.

Example 3.5

Determine the z transform of a *finite duration* sequence given below:

$$x[n] = a^n u[n] - a^n u[n-N]$$

Figure 3.10: A finite duration sequence $x[n]$.

Solution:

$$\begin{aligned} X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n &= \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} \\ &= \frac{z - \frac{a^N}{z^{N-1}}}{z-a} \\ &= \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z-a} \end{aligned}$$

(a) ROC:

$$\begin{aligned} \sum_{n=0}^{N-1} \left|az^{-1}\right|^n < \infty \quad &\longrightarrow \quad \left|az^{-1}\right| < \infty \\ &\longrightarrow \quad \frac{|a|}{|z|} < \infty \\ &\longrightarrow \quad |a| < \infty \text{ and } z \neq 0 \\ &\xrightarrow{\text{i.e.}} \quad \text{entire } z \text{ plane except } z = 0 \end{aligned}$$

(b) Poles and zeros:

(i) zeros [7] : $z^N - a^N = 0 \longrightarrow z_k = ae^{j\frac{2\pi k}{N}}, \quad k = 1, 2, \ldots, N-1$

(ii) poles: $z = 0 \quad : (N-1)st$ order

[7] Note that the term $(z-a)$ cencels out in the numerator and the denominator of $X(z)$.

Figure 3.11: The ROC of the z transform for a finite duration sequence $x[n]$.

Table 3.1: Common z-transform pairs at page 104
: *Self study (assignment).*

3.3 Properties of ROC

$$X(z) = \frac{P(z)}{Q(z)}$$

1. ROC is a *ring* or a *disc* in the z-plane centered at the origin, i.e.

$$\text{ROC} = \{z | 0 \leq r_R < |z| < r_L < \infty\}$$

2. $F\{x[n]\}$ exists if and only if ROC contains the unit circle.

3. ROC cannot contain any poles. [8]

4. If $x[n]$ is a finite duration sequence, then ROC is the entire z-plane except possibly at $z = 0$ or $z = \infty$.

   (cf.) Note that the followings:

$$|X(z)| \left| \sum_{n=N_1}^{N_2} x[n] z^{-n} \right| \leq \sum_{n=N_1}^{N_2} |x[n]| \cdot |z|^{-n}$$

   (i) if $n < 0$, then $|X(z)| \rightarrow \infty$ as $z :\rightarrow \infty$.
   (ii) if $n \geq 0$, then $|X(z)| \rightarrow \infty$ as $z :\rightarrow 0$.

5. If $x[n]$ is a *right-sided* sequence, then ROC extends outward from the outermost finite pole to $z = \infty$.

Figure 3.12: The ROC of a right-sided sequence.

[8] This is so because if ROC contains a pole, $|X(z)| \rightarrow \infty$.

6. If $x[n]$ is a *left-sided* sequence, then ROC extends inward from the innermost non-zero pole to $z = 0$.

Figure 3.13: The ROC of a left-sided sequence.

7. If $x[n]$ is a *two-sided* sequence, then ROC consists of a ring bounded by poles.

Figure 3.14: The ROC of a two-sided sequence.

8. ROC must be a connected region(i.e. cannot be disjoint).

Proof(detailed): Assignment(self study)

Example 3.6

Given a pole-zero diagram with its pole locations as follows:

Figure 3.15: An example of pole-zero diagram.

Then, there ∃ 4 possible cases of ROC depending on which we get different discrete-time signals.

(a) Right-sided sequence:

Figure 3.16: A pole-zero diagram for a right-sided sequence.

(b) Left-sided sequence:

Figure 3.17: A pole-zero diagram for a left-sided sequence.

(c) Two-sided sequence: #1

Figure 3.18: A pole-zero diagram for a two-sided sequence.

(d) Two-sided sequence: #2

Figure 3.19: Another pole-zero diagram for a two-sided sequence.

Note:

1. $x[n]$ cannot be a finite duration sequence with the given pole-zero diagram of $X(z)$.
2. The only case when $\exists$ the DTFT of $x[n]$ is (3). Why?

3.4 The inverse z-transform

EXAMPLE: Analysis of a DLTI system:

Figure 3.20: A DLTI system.

After analyzing DLTI system in z-domain (i.e. finding the output $Y(z)$), we need to compute $y[n]$ (i.e. $Y(z) \rightarrow y[n]$).

3.4.1 Inspection method

Use of familiar z-transform pairs, where tables of z-transform pairs are quite useful!!!

Example 3.7

Recall the following z-transform pair(either from the table or from the previous example ...)

$$a^n u[n] \quad \longleftrightarrow \quad \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Therefore,if we are given that:

$$X(z) = \frac{1}{1 - (\frac{1}{5})z^{-1}}, \quad |z| > \frac{1}{5}|$$

then by inspection, we get the inverse z-transform easily as:

$$x[n] = \left(\frac{1}{5}\right)^n u[n]$$

(cf.) If the ROC was $|z| < \frac{1}{5}$, then we know by inspection (from previous experiences) that:

$$x[n] = -\left(\frac{1}{5}\right)^n u[-n-1]$$

3.4.2 Partial fraction expansion: w/ inspection

Suppose $X(z)$ is in the form of *ratio of polynomials* in z^{-1}, i.e.:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{3.1}$$

$$= \frac{b_0 \prod_{k=1}^{M}(1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$\overset{\text{or}}{=} \frac{b_0}{a_0} \frac{z^N}{z^M} \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{N}(z - d_k)} \tag{3.2}$$

$$= \begin{cases} \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}, & \text{if } M < N \\ \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}}, & \text{if } M \geq N \end{cases}$$

where

(i) B_r: by long division of the numerator by the denominator

(ii) $A_k = (1 - d_k z^{-1}) X(z)|_{z=d_k}$

The inverse z-transform can then be found by:

$$x[n] = \sum_{r=0}^{M-N} \mathcal{Z}^{-1}\left\{B_r z^{-r}\right\} + \sum_{k=1}^{N} \mathcal{Z}^{-1}\left\{\frac{A_k}{1 - d_k z^{-1}}\right\}$$

where

(i) [9]

$$\mathcal{Z}^{-1}\left\{B_r z^{-r}\right\} = B_r \delta[n - r]$$

(ii) [10]

$$\mathcal{Z}^{-1}\left\{\frac{A_k}{1 - d_k z^{-1}}\right\} = \begin{cases} A_k (d_k)^n u[n], & \text{ROC: } |z| > |d_k| \\ -A_k (d_k)^n u[-n-1], & \text{ROC: } |z| < |d_k| \end{cases}$$

[9]Note that the z-transform of an unit sample sequence is: $\mathcal{Z}\{\delta[n]\} \; \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$, and thus $\mathcal{Z}\{\delta[n-n_0]\} \; \sum_{n=-\infty}^{\infty} \delta[n-n_0] z^{-n} = z^{-n_0}$.

[10]This is easily done by inspection!

Remarks:

1. When $X(z)$ has multiple poles(d_i) of order s, i.e. if $X(z)$ is in the following form:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{p(z)}{(1 - d_i z^{-1})^s q(z)}$$

Then, the partial fraction expansion of $X(z)$ is in the form given below: [11]

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_i z^{-1})^m}$$

where [12]

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}} \left\{ \frac{d^{s-m}}{dw^{s-m}} \left[(1 - d_i w)^s X(w^{-1}) \right] \right\}_{w=d_i^{-1}}, \qquad (w \triangleq z^{-1})$$

2. $X(z)$ has the same number of poles and zeros(see (3.2)), which is:

$$\# \text{ of poles and/or zeros} = \begin{cases} M, & \text{if } M > N \\ N, & \text{if } M < N \end{cases}$$

[11]The first term is only when $M > N$, and the second term represents the *single poles*, while the last term represents the *multiple poles* in $X(z)$.

[12]Note that if $s = 1$, then $C_m = A_m$.

Example 3.8

Find the inverse z-transform of $X(z)$ given below along with its ROC.

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{(1 - z^{-1})(1 - \frac{1}{2}z^{-1})}$$

where its ROC is as follows:

$$\text{ROC} = \{z||z| > 1\} \quad \Longrightarrow \quad \text{right-sided sequence}$$

Figure 3.21: The ROC of $X(z)$ with its pole-zero locations.

Solution:

Applying the partial fraction method, $X(z)$ must be in the following form: $(M = N = 2)$

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

where

(i) $B_0 = 2$ (the ratio of coefficient for z^2 or z^{-2}, i.e. the highest order.)

(ii) $A_1 = X(z)(1 - \frac{1}{2}z^{-1})\Big|_{z=\frac{1}{2}} = \frac{(1+z^{-1})^2}{1-z^{-1}}\Big|_{z=\frac{1}{2}} = \frac{9}{-1} = -9$

(ii) $A_1 = X(z)(1 - z^{-1})|_{z=1} = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}}\Big|_{z=1} = \frac{9}{\frac{1}{2}} = 8$

Therefore:

$$\begin{aligned} X(z) &= 2 - 9\frac{1}{1 - \frac{1}{2}z^{-1}} + 8\frac{1}{1 - z^{-1}} \\ \Longrightarrow x[n] &= 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n] \quad \text{(by inspection)} \end{aligned}$$

Remark: Depending on the ROC, we could have different sequences, i.e.

$$X(z) = 2 - 9\frac{1}{1 - \frac{1}{2}z^{-1}} + 8\frac{1}{1 - z^{-1}}$$

(1) ROC $= \{z||z| > 1\}$: outside of the unit circle (as in the example above)

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n] \quad : \text{right-sided sequence}$$

Figure 3.22: ROC $= \{z||z| > 1\}$: outside of the unit circle.

(2) ROC $= \{z||z| < \frac{1}{2}\}$: inside of a circle

$$x[n] = 2\delta[n] + 9\left(\frac{1}{2}\right)^n u[-n-1] - 8u[-n-1] \quad : \text{left-sided sequence}$$

Figure 3.23: ROC $= \{z||z| < \frac{1}{2}\}$: inside of a circle.

(3) ROC $= \{z|\frac{1}{2} < |z| < 1\}$: in-between two circles

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] - 8u[-n-1] \quad : \text{two-sided sequence}$$

Figure 3.24: ROC $= \{z|\frac{1}{2} < |z| < 1\}$: in-between two circles.

3.4.3 Power series expansion

Note that the definition of the z-transform $X(z)$ itself is in the form of a power series, i.e.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \ldots\ldots + x[-2]z^{2} + x[-1]z^{1} + x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots\ldots \end{aligned}$$

$\Longrightarrow$ Finding $x[n]$ is equivalent to determining the coefficients of z^{-n} in $X(z)$!!!

Example 3.9

Find the inverse z-transform of $X(z)$ given below, where the ROC is the entire z-plane except at $z = 0$.

$$X(z) = \frac{(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1})}{z^{-2}}$$

Solution:

Developing the given $X(z)$, we get:

$$\begin{aligned} X(z) &= z^{2}(1 - \frac{1}{2}z^{-1})(1 + z^{-1})(1 - z^{-1}) \\ &= 1 \cdot z^{2} - \frac{1}{2} \cdot z - 1 + \frac{1}{2} \cdot z^{-1} \\ &= x[-2] \cdot z^{2} + x[-1] \cdot z + x[0] + x[1] \cdot z^{-1} \end{aligned}$$

Therefore,

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

(cf.) Find $x[n]$ using the partial fraction expansion method: *assignment*

Example 3.10

Find the inverse z-transform of $X(z)$ given below, where the ROC is the outside of a circle with radius $|a|$.

$$X(z) = \log(1 + az^{-1}), \qquad \text{ROC} = \{z||z| > |a|\}$$

Solution:

Developing the given $X(z)$ using the logarithmic series expansion [13], we get:

$$\begin{aligned} X(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^n}{n}, \quad |az^{-1}| < 1 \text{ (i.e. } |z| > |a|) \\ &\triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n} \end{aligned}$$

Therefore:

$$x[n] = \left\{ \begin{array}{ll} \frac{(-1)^{n+1}a^n}{n}, & n \geq 1 \\ 0, & n \leq 0 \end{array} \right\} = \frac{(-1)^{n+1}a^n}{n} u[n]$$

(cf.)

Note that $x[n]$ is a right-sided sequence, since the ROC is given as the outside of a circle.

Example 3.11

Find the inverse z-transform of $X(z)$ given below, which we already have discussed in previous examples [14] , using the power series expansion method.

$$X(x) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

[13] Logarithmic series: $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$, where $|x| < 1$.

[14] We know the answer as: $x[n] = a^n u[n]$ if $|z| > |a|$, and $x[n] = -a^n u[-n-1]$ if $|z| < |a|$.

Solution:

There ∃ two possible ROC's for the given $X(z)$:

(i) ROC: $|z| > |a|$ (i.e. right-sided sequence)

Since $x[n]$ must be a right-sided sequence, $X(z)$ should be expressed as a series in powers of z^{-1} ($\because\ n \geq 0$)

$\Longrightarrow$ By long division, we get:

$$X(z) = \frac{1}{1 - az^{-1}} \quad = \quad \ldots\ldots$$

$$= \quad \ldots\ldots$$

$$= \quad \vdots$$

$$X(z) = \frac{1}{1 - az^{-1}} \quad = \quad 1 + az^{-1} + a^2z^{-2} + \ldots$$

$$= \quad x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

Therefore, we have:

$$x[n] = a^n u[n]$$

(ii) ROC: $|z| < |a|$ (i.e. left-sided sequence)

Since $x[n]$ must be a left-sided sequence, $X(z)$ should be expressed as a series in powers of z ($\because\ n < 0$)

$\Longrightarrow$ By long division, we get:

$$X(z) = \frac{1}{-az^{-1} + 1} \quad = \quad \ldots\ldots$$

$$= \quad \ldots\ldots$$

$$= \quad \vdots$$

$$X(z) = \frac{1}{-az^{-1} + 1} \quad = \quad -a^{-1}z - a^{-2}z^2 + a^{-3}z^3 + \ldots$$

$$= \quad x[-1]z + x[-2]z^2 + x[-3]z^3 + \ldots$$

Therefore, we have:

$$x[n] = -a^n u[-n-1]$$

3.5 The z-transform properties

Let

$$X(z) = \mathcal{Z}\{x[n]\}, \qquad \text{ROC} = R_x$$

$$X_1(z) = \mathcal{Z}\{x_1[n]\}, \qquad \text{ROC} = R_{x_1}$$

$$X_2(z) = \mathcal{Z}\{x_2[n]\}, \qquad \text{ROC} = R_{x_2}$$

(1) Linearity:

$$\mathcal{Z}\{ax_1[n] + bx_2[n]\} = aX_1(z) + bX_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

proof: assignment (trivial)

NOTE:
The fact that ROC $\supseteq R_{x_1} \cap R_{x_2}$ rather than ROC $= R_{x_1} \cap R_{x_2}$ is **due to the possible cancellation of poles in** $X(z)$.

Example 3.12

Consider the finite duration sequence $x[n]$ discussed in the previous example:

$$\begin{aligned} x[n] &= a^n u[n] - a^n u[n-N] \\ &= x_1[n] - x_2[n] \end{aligned}$$

We already know that the ROC's each sequence are as follows;

$$\begin{cases} R_{x_1}: & |z| > |a| \\ R_{x_2}: & |z| > |a| \\ R_x: & \text{entire z-plane except at} z = 0 \end{cases}$$

Figure 3.25: The ROC of a finite duration sequence as $R_x \supset R_{x_1} \cap R_{x_2}$.

Note that $R_x \supset R_{x_1} \cap R_{x_2}$, and this results from the cancellation of the term $1 - az^{-1}$ in the numerator and the denominator of $X(z)$, i.e.

$$\begin{cases} X_1(z) = \frac{1}{1-az^{-1}} \\ \\ X_2(z) = \sum_{n=N}^{\infty} a^n z^{-n} = \sum_{n=N}^{\infty} (az^{-1})^n = \frac{(az^{-1})^N}{1-az^{-1}} \end{cases}$$

Thus;

$$X(z) = X_1(z) - X_2(z) = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1 - az^{-1}}{1 - az^{-1}} \cdot q(z)$$

where the term $1 - az^{-1}$ cancels out which eliminates the pole located at $z = a$, and corresponding ROC extends to the origin.

(2) Time shifting:

$$\mathcal{Z}\{x[n - n_0]\} = X(z)z^{-n_0}, \qquad \text{ROC} = R_x \pm \{z = 0 \text{ or } z = \infty\}$$

proof: assignment (trivial)

NOTE:
The fact that ROC $= R_x \pm \{z = 0$ or $z = \infty\}$ is **due to added term** z^{-n_0} by which $z = 0$ and $z = \infty$ arises for $n_0 < 0$ and $n_0 > 0$ respectively.

Example 3.13

Find the inverse z-transform of the following $X(z)$:

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}}, \qquad |z| > \frac{1}{4} \quad \text{(right sided sequence)}$$

Solution:

We will use two different appoaches to obtain $x[n]$:

(a) Ordinary way:

By applying the partial fraction expansion, we get [15]

$$X(z) = -4 + \frac{4}{1 - \frac{1}{4}z^{-1}}$$

Therefore, by inspection we obtain:

$$\begin{aligned} x[n] &= -4\delta[n] + 4\left(\frac{1}{4}\right)^n u[n] \\ &= 4\left(\frac{1}{4}\right)^n u[n-1] \\ &= \left(\frac{1}{4}\right)^{n-1} u[n-1] \end{aligned}$$

(b) Utilizing the time-shift property:

Express $X(z)$ in the following form:

$$X(z) = z^{-1} \cdot \left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right)$$

Then, $x[n]$ can be obtained as:

$$\begin{aligned} x[n] &= \mathcal{Z}^{-1}\left\{\frac{1}{1 - \frac{1}{4}z^{-1}}\right\}\Bigg|_{n \to n-1} \\ &= \left(\frac{1}{4}\right)^n u[n]\Bigg|_{n \to n-1} \\ &= \left(\frac{1}{4}\right)^{n-1} u[n-1] \end{aligned}$$

which is the same result as in (a)!!!

[15]By partial fraction expansion, $X(z) = -4 + \frac{A_1}{1-\frac{1}{4}z^{-1}}$, where $A_1 = z^{-1}\big|_{z=\frac{1}{4}} = 4$.

(3) Multiplication by an exponential sequence:

$$\mathcal{Z}\left\{x[n]z_0^n\right\} = X\left(\frac{z}{z_0}\right), \qquad \text{ROC} = R_x \cdot |z_0|$$

proof: assignment

Remarks:

(1) If $R_x = \{z| \; r_R < |z| < r_L\}$, then the ROC of $x[n]z_0^n$ becomes:

$$\begin{aligned}\text{ROC} &= \left\{z| \; r_R < \left|\frac{z}{z_0}\right| < r_L\right\} \\ &= \{z| \; |z_0|r_R < |z| < |z_0|r_L\}\end{aligned}$$

(2) Pole-zero locations are also scaled by the factor of z_0, i.e. the location z_1 in $X(z)$ becomes the location $z_0 z_1$ in $X\left(\frac{z}{z_0}\right)$. [16]

Special Cases:

(i) If z_0 is a positive real number:

Only magnitude changes, which means that pole and/or zero moves in *radial direction*!

(ii) If z_0 is complex w/ unit magnitude (i.e. $z_0 = e^{j\omega_0}$):

Pole and/or zero *rotates by an angle of* ω_0, which means that **frequency shift** occurs! [17]

i.e.:

$$e^{j\omega_0 n}x[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X\left(\frac{e^{j\omega}}{e^{j\omega_0}}\right) = X\left(e^{j(\omega-\omega_0)}\right)$$

---

[16] The term $(z - z_1)$ in $X(z)$, whose root is $z = z_1$, is being transformed into a term $\left(\frac{z}{z_0} - z_1\right)$ in $X\left(\frac{z}{z_0}\right)$ where corresponding root then becomes $\frac{z}{z_0} = z_1$; that is $z = z_0 z_1$.

[17] Recall the frequency shift property of the DTFT, that is $e^{j\omega_0 n}x[n] \overset{F}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$ if there $\exists$ $X(e^{j\omega})$.

Example 3.14

Recall that the z-transform of the unit step sequence is as follows:

$$u[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{1-z^{-1}}, \quad |z| > 1$$

Then, find the z-transform of the exponentially decaying (or growing) sinusoidal sequence given below:

$$x[n] = r^n \cos(\omega_0 n) u[n]$$

Solution:

Express $x[n]$ as:

$$\begin{aligned} x[n] &= r^n \cos(\omega_0 n) u[n] \\ &= \frac{1}{2}(re^{j\omega_0})^n u[n] + \frac{1}{2}(re^{-j\omega_0})^n u[n] \\ &\triangleq x_1[n] + x_2[n] \end{aligned}$$

Then, we have:

$$X_1(z) = \frac{1}{2} U\left(\frac{z}{re^{j\omega_0}}\right) = \frac{1}{2} \frac{1}{1 - re^{j\omega_0} z^{-1}}$$

where corresponding ROC of $X_1(z)$ becomes: $|z| > 1 \cdot |re^{j\omega_0}| = r$.

And

$$X_2(z) = \frac{1}{2} U\left(\frac{z}{re^{-j\omega_0}}\right) = \frac{1}{2} \frac{1}{1 - re^{-j\omega_0} z^{-1}}$$

where corresponding ROC of $X_2(z)$ becomes: $|z| > 1 \cdot |re^{-j\omega_0}| = r$.

Therefore, the z-transform of $x[n]$ is then,

$$X(z) = X_1(z) + X_2(z) = \frac{1 - r\cos(\omega_0) z^{-1}}{1 - 2r\cos(\omega_0) z^{-1} + r^2 z^{-2}}, \quad \text{ROC} = \{z| \ |z| > r\}$$

(4) Convolution of sequences:

$$\mathcal{Z}\{x_1[n] * x_2[n]\} = X_1(z) \cdot X_2(z), \qquad \text{ROC} \supseteq R_{x_1} \cap R_{x_2}$$

proof: assignment

Remarks:

(1) The fact that ROC $\supseteq R_{x_1} \cap R_{x_2}$ rather than ROC $= R_{x_1} \cap R_{x_2}$ is again **due to the possible cancellation of poles in** $X(z)$.

(2) This property is very useful in the *analysis* of a DLTI system.

(e.g.)

Figure 3.26: A DLTI system.

$$y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z)$$

where

$$H(z) = \frac{Y(z)}{X(z)} \quad : \text{system function}$$

Example 3.15

Determine the output sequence of the *accumulator* when the input signal is an exponentially decaying sequence, i.e.

$$h[n] = u[n]$$

$$x[n] = a^n u[n], \quad \text{where } 0 < a < 1$$

Solution:

We can obtain the output $y[n]$ by taking convolution sum b/w $h[n]$ and $x[n]$ (*assignment*), which might be very cumbersome to do!!! Instead, we try to get the output in z-domain:

We already know that

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$H(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

Therefore, from the convolution property of z-transform;

$$Y(z) = H(z) \cdot X(z) = \frac{1}{1 - z^{-1}} \cdot \frac{1}{1 - az^{-1}} = \frac{z^2}{(z-a)(z-1)}$$

where the ROC of $Y(z)$ is

$$\text{ROC} = R_y = \{z \mid |z| > 1\}, \quad \text{since } \ |a| < 1$$

Figure 3.27: The ROC R_y of the output signal w/ its pole-zero locations.

Taking the partial fraction expansion of $Y(z)$, we get;

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1 - z^{-1}} - \frac{a}{1 - az^{-1}}\right), \quad |z| > 1$$

Therefore, by taking the inverse z-transform of $Y(z)$, we obtain

$$y[n] = \mathcal{Z}^{-1}\{Y(z)\} = \frac{1}{1-a}\left(u[n] - a^{n+1}u[n]\right) = \frac{1}{1-a}\left(1 - a^{n+1}\right)u[n]$$

(5) Initial value theorem:

If $x[n] = 0 \ \forall\, n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

proof: assignment (problem 3.54 at your testbook)

OTHER PROPERTIES: *Self Study*

(6) Differentiation of $X(z)$: at p.122

(7) Conjugate of complex sequence: at p.123

(8) Time reversal: at p.123

SUMMARY (Table 3.2): *Self Study*

3.6 The inverse z-transform using contour integration *:Formal expression for inverse z-transform*

Cauchy Integral Theorem(Formula): [18]

$$\frac{1}{2\pi j}\oint_C z^{-k}dz = \left\{ \begin{array}{ll} 1, & k = 1 \\ \\ 0, & k \neq 1 \end{array} \right\} = \delta[k-1]$$

where C is a CCW(counter clockwise) contour encircling the origin. [19]

Figure 3.28: Cauchy residue theorem: integrating z^{-k} over a CCW contour C in z-plane..

Derivation of inverse z-transform:

¿From the z-transform formula:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Multiplying z^{k-1} to both sides and integrating over a CCW contour encircling the origin *within the ROC of* $X(z)$, we get:

$$\begin{aligned}
\frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz &= \frac{1}{2\pi j}\oint_C \sum_{n=-\infty}^{\infty} x[n]z^{-n+k-1}dz \\
&= \sum_{n=-infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_C z^{-n+k-1}dz \\
&= \sum_{n=-infty}^{\infty} x[n]\frac{1}{2\pi j}\oint_C z^{-(n-k+1)}dz \\
&= \sum_{n=-\infty}^{\infty} x[n]\delta[n-k] \quad \text{(by Cauchy integral theorem)} \\
&= x[k]
\end{aligned}$$

[18] Line integral or contour integral

[19] This will be officially proved using the Residue theorem at later section.

Therefore, the inverse transform $x[n]$ of $X(z)$ in terms of countour integration can be expressed in the following formula:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{3.3}$$

where C is a CCW contour encircling the origin within the ROC.

Remarks:

1. If the integration contour C is taken to be the unit circle (i.e. $z = e^{j\omega}$), (3.3) reduces to be the inverse DTFT, i.e.

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Let:

(i) $z = e^{j\omega} \quad \longrightarrow$ contour C in z-plane becomes an interval $\omega = [-\pi, \pi]$.

(ii) $dz = j e^{j\omega} d\omega$

Therefore,

$$\begin{aligned} x[n] &= \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n - j\omega} \cdot j e^{j\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \\ &: \quad \text{inverse DTFT} \end{aligned}$$

2. (3.3) can be evaluated by the Cauchy Residue Theorem, which is:

$$\begin{aligned} x[n] &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \\ &= \sum \{\text{residues of } X(z) z^{n-1} \textbf{ at the poles inside } C\} \end{aligned}$$

where if the integrand is a rational function of z, i.e.

$$X(z)z^{n-1} = \frac{\psi(z)}{(z-d_0)^s}$$

then,

$$\text{Res}\left[X(z)z^{n-1} \text{ at } z = d_0\right] = \frac{1}{(s-1)!} \left.\frac{d^{s-1}\psi(z)}{dz^{s-1}}\right|_{z=d_0}$$

(cf.) If $s = 1$ (single pole), then $\text{Res}\,[X(z)z^{n-1} \text{ at } z = d_0] = \psi(d_0)$, assuming $z = d_0$ is located inside of C.

3. **Proof of Cauchy integral theorem:**
Applying the Cauchy residue theorem, we get:

$$\begin{aligned} \frac{1}{2\pi j}\oint_C z^{-k}dz &= \begin{cases} 0, & k \le 0 \quad (\because \text{ no ploes}) \\ 1, & k = 1 \quad (\because \text{ single ploe at } z = 0) \\ 0, & k > 1 \quad (\because \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\{(1)\} = 0) \end{cases} \\ &= \delta[k-1] \end{aligned}$$

Example 3.16

Find the inverse z-transform of $X(z)$ given below: [20]

$$X(z) = \frac{1}{1-az^{-1}}, \quad \text{ROC: } |z| > |a|$$

Solution:

Using the formal expression of the inverse z-transform,

$$\begin{aligned} x[n] &= \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \\ &= \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}}dz \\ &= \frac{1}{2\pi j}\oint_C \frac{z^n}{z-a}dz \end{aligned}$$

where C is taken to be *a circle of radius greater than* $|a|$ (i.e. a contour *within ROC* encircling the origin).

[20] We already know from previous examples that $\mathcal{Z}^{-1}\{X(z)\} = x[n] = a^n u[n]$.

Figure 3.29: The integration contour C in z-plane.

(1) $n \geq 0$: (a single pole at $z = a$: inside of C)

$$\begin{aligned} x[n] &= \sum [\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C] \\ &= z^n\big|_{z=a} \\ &= a^n \end{aligned}$$

(2) $n < 0$: (multiple poles at $z = 0$ & a single pole at $z = a$: inside of C)

$$x[n] = \sum [\text{residues of } X(z)z^{n-1} \text{ at the poles inside } C]$$

(i) $n = -1$:

$$\begin{aligned} x[-1] &= \sum [\text{residues of } X(z)z^{-2} \text{ at the poles inside } C] \\ &= \sum \left[\text{residues of } \tfrac{1}{z(z-a)} \text{ at } z = 0 \ \& \ z = a\right] \\ &= -\frac{1}{a} + \frac{1}{a} \\ &= 0 \end{aligned}$$

(ii) $n = -2$:

$$\begin{aligned}
x[-2] &= \sum [\text{residues of } X(z)z^{-3} \text{ at the poles inside } C] \\
&= \sum \left[\text{residues of } \tfrac{1}{z^2(z-a)} \text{ at } z = 0 \text{ \& } z = a\right] \\
&= \frac{1}{1!}\frac{d}{dz}\left(\frac{1}{z-a}\right)\bigg|_{z=0} + \frac{1}{z^2}\bigg|_{z=a} \\
&= \frac{-1}{(z-a)^2}\bigg|_{z=0} + \frac{1}{a^2} \\
&= -\frac{1}{a^2} + \frac{1}{a^2} \\
&= 0
\end{aligned}$$

$$\vdots$$

(tedius to carry out!!!)

Likewise, we get $x[n] = 0 \;\; \forall\, n < 0$, and therefore:

$$x[n] = a^n u[n]$$

(cf.) For the case of $n < 0$, let $m = -n$, thus making $m > 0$, then: [21]

$$\begin{aligned}
x[n] = x[-m] &= \frac{1}{2\pi j}\oint_C \frac{1}{(z-a)z^m}dz \\
&= \sum \left[\text{residues of } \tfrac{1}{(z-a)z^m} \text{ at } z = a \text{ \& } z = 0\right] \\
&= \frac{1}{z^m}\bigg|_{z=a} + \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{\frac{1}{z-a}\right\}\bigg|_{z=0} \\
&= \frac{1}{a^m} + \frac{1}{(m-1)!}\frac{(-1)^{m-1}(m-1)!}{(z-a)^m}\bigg|_{z=0} \\
&= \frac{1}{a^m} + \frac{1}{(m-1)!}\frac{(-1)^{m-1}(m-1)!}{(-1)^m a^m} \\
&= \frac{1}{a^m} - \frac{1}{a^m} \\
&= 0
\end{aligned}$$

[21] Let $f(z) = \frac{1}{z-a}$, then $f^{(n)}(z) = \frac{(-1)^n n!}{(z-a)^{n+1}}$.

Remark:

The inverse z-transform formula (3.3) is very cumbersome to carry out for the case when $n < 0$, since we get *multiple poles* at $z = 0$ due to the factor z^{n-1} in the integrand(see below).

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

This can be avoided by the change of variable technique, i.e. by letting:

$$z = p^{-1}$$

we get an equivalent formula of: [22]

$$\begin{aligned} x[n] &= \frac{1}{2\pi j} \oint_{C''} X\left(\frac{1}{p}\right) p^{-n-1} dp \\ &= \sum \text{Res}\left[X\left(\frac{1}{p}\right) p^{-n-1} \text{ at poles inside of } C''\right] \end{aligned}$$

where C'' is a **CCW** circle of radius less tha $\frac{1}{r}$, if C was a CCW circle of radius greater than r.

Note:

(1) The integration contour is now CCW by exchanging the sign of the integration and the direction of the contour!!! (i.e. $-p^{-1}dp \rightarrow p^{-2}dp$ makes the CW contour C' a CCW contour C'')

(2) The above formula for inverse z-transform, on the contrary, will cause multiple poles at $p = 0$ when $n \geq 0$.

proof: done (refer the footnote below.)

[22]Note that from $z = p^{-1}$ we have: $dz = -p^{-2}dp$, $z^{n-1} = p^{-n+1}$, and the CCW contour C on z becomes a CW(clockwise) contour C' on p.

Example 3.17

Redo the previous example for the case of $n < 0$.

Solution:

Figure 3.30: The CCW integration contour C' on the p plane.

$$\begin{aligned} x[n] &= \frac{1}{2\pi j} \oint_{C'} \frac{p^{-n-1}}{1-ap} dp \quad (C': \text{ radius of less than } \tfrac{1}{a}) \\ &= \sum \text{Res}\left[\frac{p^{-n-1}}{1-ap} \text{ at poles inside of } C'\right] \quad \text{(NONE)} \\ &= 0 \end{aligned}$$

3.7 The complex convolution theorem

: Relative to (or generalization of) the *periodic convolution property* of DTFT

(cf.) Periodic convolution property of DTFT(Recall from S&S class)
:Windowing theorem or modulation property

Let $w[n] = x_1[n] \cdot x_2[n]$, then

$$\begin{aligned} \mathrm{F}\{w[n]\} = W\left(e^{j\omega}\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(e^{j\Omega}\right) X_2\left(e^{j(\omega-\Omega)}\right) d\Omega \\ &\triangleq \frac{1}{2\pi} X_1\left(e^{j\omega}\right) \otimes X_2\left(e^{j\omega}\right) \end{aligned}$$

Theorem 3.1 Let $w[n] = x_1[n] \cdot x_2[n]$, then the z-transform $W(z)$ of $w[n]$ is in the following form:

$$W(z) = \frac{1}{2\pi j}\oint_{C_2} X_1\left(\frac{z}{v}\right) X_2(v) v^{-1} dv$$

where C_2 is a CCW contour within the overlap of ROC R_{x_2} of $X_2(v)$ and ROC of $X_1\left(\frac{z}{v}\right)$.

OR,

$$W(z) = \frac{1}{2\pi j}\oint_{C_1} X_1(v) X_2\left(\frac{z}{v}\right) v^{-1} dv$$

where C_1 is a CCW contour within the overlap of ROC R_{x_1} of $X_1(v)$ and ROC of $X_2\left(\frac{z}{v}\right)$.

Derivation:

Since

$$w[n] = x_1[n] \cdot x_2[n]$$

we have:

$$W(z) \triangleq \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]z^{-n} \tag{3.4}$$

Here,

$$x_2[n] = \frac{1}{2\pi j} \oint_{C_2} X_2(v)v^{n-1}dv \tag{3.5}$$

where C_2 is a CCW contour within R_{x_2}.

Inserting (3.5) into (3.4), we get:

$$\begin{aligned} W(z) &= \frac{1}{2\pi j} \sum_{n=-\infty}^{\infty} x_1[n] \oint_{C_2} X_2(v) \left(\frac{z}{v}\right)^{-n} v^{-1}dv \\ &= \frac{1}{2\pi j} \oint_{C_2} \left\{ \sum_{n=-\infty}^{\infty} x_1[n] \left(\frac{z}{v}\right)^{-n} \right\} X_2(v)v^{-1}dv \\ &= \frac{1}{2\pi j} \oint_{C_2} X_1\left(\frac{z}{v}\right) X_2(v)v^{-1}dv \end{aligned} \tag{3.6}$$

where C_2 should bow be a CCW contour within the overalp of ROC of $X_1\left(\frac{z}{v}\right)$ and ROC of $X_2(v)$.

Remark:

1. ROC R_w of $W(z)$:

   Let

$$R_{x_1} \; : \quad r_{R_1} < |z| < r_{L_1}$$

$$R_{x_2} \; : \quad r_{R_2} < |z| < r_{L_2}$$

Then, from (3.6), the contour C_2 is within regions of:

$$\text{(i)}\ \ X_2(v)\ :\quad r_{R_2} < |v| < r_{L_2}$$

$$\text{(ii)}\ \ X_1\left(\frac{z}{v}\right)\ :\quad r_{R_1} < \left|\frac{z}{v}\right| < r_{L_1}$$

From (ii), we have $r_{R_1}|v| < |z| < r_{L_1}|v|$, and combining (i) and (ii) we get the ROC R_w of $W(z)$ as:

$$r_{R_1} r_{R_2} < |z| < r_{L_1} r_{L_2}$$

$\Rightarrow$ We denote it as $R_w = R_{x_1} \cdot R_{x_2}$, but notice that R_w may actually be larger than $R_{x_1} \cap R_{x_2}$, depending on possible cancellation of poles.

2. Periodic convolution of DTFT:

In (3.6), let C_2 (and/or C_1) be the unit circle(s), which means the change of variable as $v = e^{j\Omega}$, then:

(i) $C_2 \quad \longrightarrow \quad -\pi \leq \Omega \leq \pi$

(ii) $dv = je^{j\Omega} d\Omega$

Also, let $z = e^{j\omega}$, then (3.6) becomes the DTFT $W\left(e^{j\omega}\right)$ of $w[n]$:

$$\begin{aligned}
W\left(e^{j\omega}\right) &= \frac{1}{2\pi j}\int_{-\pi}^{\pi} X_1\left(e^{j(\omega-\Omega)}\right) X_2\left(e^{j\Omega}\right) e^{-j\Omega}\cdot je^{j\Omega} d\Omega \\
&= \frac{1}{2\pi j}\int_{-\pi}^{\pi} X_1\left(e^{j(\omega-\Omega)}\right) X_2\left(e^{j\Omega}\right) d\Omega \\
&= \frac{1}{2\pi} X_1\left(e^{j\omega}\right) \otimes X_2\left(e^{j\omega}\right)
\end{aligned}$$

as we expected!!!

Example 3.18

Let $w[n] = x_1[n]x_2[n]$ where $x_1[n] = a^n u[n]$ and $x_2[n] = b^n u[n]$.
Determine the z-transform $W(z)$ of $w[n]$.

Solution:

We already have the following z-transform pairs:

$$x_1[n] = a^n u[n] \quad \longleftrightarrow \quad X_1(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$x_2[n] = b^n u[n] \quad \longleftrightarrow \quad X_2(z) = \frac{1}{1 - bz^{-1}}, \quad |z| > |b|$$

From (3.6), the z-transform of $w[n]$ is then:

$$\begin{aligned} W(z) &= \frac{1}{2\pi j} \oint_{C_2} \frac{1}{1 - a\left(\frac{z}{v}\right)^{-1}} \cdot \frac{1}{1 - bz^{-1}} v^{-1} dv \\ &= \frac{1}{2\pi j} \oint_{C_2} \frac{-\frac{z}{a}}{\left(v - \frac{z}{a}\right)} \cdot \frac{1}{v - b} dv \end{aligned}$$

Notice that:

$$\begin{cases} \text{pole } \#1: & v = b \\ \text{pole } \#2: & v = \frac{z}{a} \end{cases}$$

and, since C_2 MUST be within overlap region of the ROC's of $X_2(v)$ and $X_1\left(\frac{z}{v}\right)$, each ROC should be as follows;

$$\begin{cases} \text{ROC of } X_2(v): & |v| > |b| \\ \text{ROC of } X_1\left(\frac{z}{v}\right): & \left|\frac{z}{v}\right| > |a| \longrightarrow |v| < \frac{|z|}{|a|} \end{cases}$$

Note that pole at $v = b$ is inside of C_2 whereas pole at $v = \frac{z}{a}$ is outside of C_2. Therefore, by the Cauchy's residue theorem, we get:

$$\begin{aligned} W(z) &= \text{Res}\left[\frac{-\frac{z}{a}}{v - \frac{z}{a}} \cdot \frac{1}{v - b} \text{ at pole } v = b\right] \\ &= \frac{-\frac{z}{a}}{b - \frac{z}{a}} \\ &= \frac{1}{1 - abz^{-1}}, \qquad |z| > |ab| \end{aligned}$$

(cf.) The ROC R_w of $W(z)$:

Note from the ROC's of $X_2(v)$ and $X_1\left(\frac{z}{v}\right)$, we have:

$$
\begin{aligned}
& |b| < |v| < \frac{|z|}{|a|} \\
\rightarrow \quad & |z| > |v| \cdot |a| \ \text{ and } \ |v| > |b| \\
\rightarrow \quad & |z| > |a| \cdot |b|
\end{aligned}
$$

Figure 3.31: ROC of $X_2(v)$ and $X_1\left(\frac{z}{v}\right)$ in v-plane.

Note:

Since $w[n]$ can be put into the following form:

$$x_1[n]x_2[n] = a^n b^n u[n] = (ab)^n u[n]$$

we can directly derive the z-transform by a simple inspection as:

$$W(z) = \frac{1}{1 - abz^{-1}}, \qquad \text{ROC: } |z| > |ab|$$

3.8 The Parseval's theorem

Theorem 3.2

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

where C is a CCW contour within overlap of ROC of $X_1(v)$ and ROC of $X_2^*\left(\frac{1}{v^*}\right)$.

Derivation:

Let $y[n] = x_1[n]x_2^*[n]$, then from the complex convolution theorem, we have: [23]

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n]z^{-n} \\ &= \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{z^*}{v^*}\right)v^{-1}dv \end{aligned}$$

Put $z = 1$ in both sides [24] , then

$$Y(z)|_{z=1} = \sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

q.e.d.

[23]Note that $x*[n] \leftrightarrow X^*(z^*)$; refer to Table 3.2 at p.126 of the textbook.

[24]Since (i) $z = 1$ must be inside of R_y, and (ii) ROC is composed of circles $\Rightarrow$ ROC R_y must include the **unit circle** for the Parseval's theroem to be valid, i.e. Parseval;s theorem can only be applied to *absolutely summable* sequences whose DTFT exists.

Remarks:

(1) If $x_1[n] = x_2[n] = x[n]$ are *real* sequences, then the Parseval's theorem becomes:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi j} \oint_C X(v)X(v^{-1})v^{-1}dv$$

and it represents the *energy* in $x[n]$:

(i) LHS = energy of $x[n]$ in time domain

(ii) RHS = energy of $x[n]$ in z (or frequency) domain

(2) DTFT equivalent form:

Let $v = e^{j\omega}$, then the Parseval's theorem states:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] &= \frac{1}{2\pi j} \int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*\left((e^{-j\omega})^*\right) e^{-j\omega} j e^{j\omega} d\omega \\ &= \frac{1}{2\pi j} \int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*(e^{j\omega}) d\omega\end{aligned}$$

Example 3.19

Suppose $x[n]$ is a *right-sided* **real** sequence with its z-transform given below:

$$X(z) = \frac{1}{1 - az^{-1}} \cdot \frac{1}{1 - bz^{-1}}$$

where $0 < a < b < 1$.

Then, determine the energy contained in $x[n]$.

Solution:

From the Parseval's theorem, we have:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} y[n] &\equiv \sum_{n=-\infty}^{\infty} x^2[n] \\
&= \frac{1}{2\pi j}\oint_C X(v)X(v^{-1})v^{-1}dv \\
&= \frac{1}{2\pi j}\oint_C \frac{1}{(1-av^{-1})(1-bv^{-1})}\cdot\frac{1}{(1-av)(1-bv)}\cdot\frac{1}{v}dv \\
&= \frac{1}{2\pi j}\oint_C \frac{v^2}{(v-a)(v-b)}\cdot\frac{1}{(1-av)(1-bv)}\cdot\frac{1}{v}dv \\
&= \frac{1}{2\pi j}\oint_C \frac{v}{(v-a)(v-b)}\cdot\frac{1}{(1-av)(1-bv)}dv
\end{aligned}$$

where C is taken to be the unit circle, since the unit circle must be within R_y(refer the footnote #24).

(cf.)

(i) ROC of $X(v)$:

Figure 3.32: ROC of $X(v)$.

(ii) ROC of $X(\frac{1}{v})$:

Figure 3.33: ROC of $X(\frac{1}{v})$.

(iii) The integration contour C in v-plane with ROC R_y:

Figure 3.34: The CCW integration contour C in v-plane with ROC R_y.

Therefore, the energy in $x[n]$ is:

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} x^2[n] &= \sum \text{Res}\left\{ \frac{v}{(v-a)(v-b)(1-av)(1-bv)} \text{ at poles inside of } C \right\} \\
&= \sum \text{Res}\left\{ \frac{v}{(v-a)(v-b)(1-av)(1-bv)} \text{ at } v=a \text{ and } v=b \right\} \\
&= \frac{a}{(a-b)(1-a^2)(1-ab)} + \frac{b}{(b-a)(1-ab)(1-b^2)} \\
&= \frac{a(1-b^2)-b(1-a^2)}{(a-b)(1-a^2)(1-b^2)(1-ab)} \quad \text{(joules)}
\end{aligned}
$$

(cf.) Notice that poles at $v = \frac{1}{a}$ and $v = \frac{1}{b}$ are outside of the unit circle C.

Remark:

Evaluating the energy of $x[n]$ in time domain would be very difficult, if not impossible, i.e.:

$$
X(z) \xrightarrow{Z^{-1}} x[n] \longrightarrow \sum_{n=-\infty}^{\infty} x^2[n]
$$

Assignment: Try the procedure described above.

3.9 The unilateral z-transform

Definition 3.2 The unilateral z-transform of a sequence $x[n]$ is defined as:

$$\mathcal{X}(z) \triangleq \sum_{\mathbf{n=0}}^{\infty} x[n]z^{-n}$$

Remark:

(1) So far, we considered the so called "bi-lateral" z-transform:

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

(2) If $x[n] = 0$ for all $n < 0$, then $X(z) = \mathcal{X}(z)$.

(3) All of the *ROC properties* of $\mathcal{X}(z)$ are the same as those of $X(z)$.

(4) Some of the *properties of* $\mathcal{X}(z)$ are the same, but some are different from those of $X(z)$.

Example 3.20

Let $x[n] = \delta[n]$, then:

$$\text{(i) } X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = \delta[0]z^{0} = 1$$

$$\text{(ii) } \mathcal{X}(z) = \sum_{n=0}^{\infty} \delta[n]z^{-n} = \delta[0]z^{0} = 1$$

$$\Longrightarrow X(z) = \mathcal{X}(z)$$

Example 3.21

Let $x[n] = \delta[n+1]$, then:

$$\text{(i) } X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = \delta[0] z^{1} = z$$

$$\text{(ii) } \mathcal{X}(z) = \sum_{n=0}^{\infty} \delta[n+1] z^{-n} = 0$$

$$\Longrightarrow \; X(z) \neq \mathcal{X}(z)$$

Remark:

The principal use of $\mathcal{X}(z)$ is iin analyzing DLTI systems described by a linear constant coefficient difference equation with *non-initial* (i.e. $n \neq 0$) rest conditions.

Let $y[n] = x[n-m]$ where $m > 0$, then:

$$\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-m] z^{-n} \\
&= \underbrace{x[-m]z^{0}}_{n=0} + \underbrace{x[1-m]z^{1}}_{n=1} + \cdots + \underbrace{x[-1]z^{-m+1}}_{n=m-1} + \underbrace{x[0]z^{-m}}_{n=m} + \underbrace{x[1]z^{-m-1}}_{n=m+1} + \cdots \\
&= \sum_{k=1}^{m} x[k-1-m] z^{-k+1} + \sum_{n=0}^{\infty} x[k-m] z^{-k} \\
&\quad \text{(let } k-m=n\text{, then } k=m+n\text{)} \\
&= \sum_{n=1-m}^{0} x[n-1] z^{-n-m+1} + \sum_{n=0}^{\infty} x[n] z^{-n} z^{-m} \\
&= \sum_{n=1-m}^{0} x[n-1] z^{-n-m+1} + \mathcal{X}(z) z^{-m} \\
&\quad \text{(let } n-1 \;\rightarrow\; n\text{)} \\
&= \sum_{n=-m}^{-1} x[n] z^{-n-m} + \mathcal{X}(z) z^{-m}
\end{aligned}$$

OR

$$
\begin{aligned}
\mathcal{Y}(z) &= \sum_{n=0}^{\infty} x[n-m]z^{-n} \\
& \quad (\text{let } n-m=k) \\
&= \sum_{k=-m}^{\infty} x[k]z^{-k-m} \\
&= \sum_{k=-m}^{-1} x[k]z^{-k-m} + \left(\sum_{k=0}^{\infty} x[k]z^{-k}\right) z^{-m} \\
& \quad (\text{let } k \ \rightarrow \ n) \\
&= \sum_{n=-m}^{-1} x[n]z^{-n-m} + \mathcal{X}(z)z^{-m}
\end{aligned}
$$

Note:
Notice that the time shift property of $\mathcal{X}(z)$ is different from that of $X(z)$!!!

Example 3.22

Given a DLTI system with the i/o relation of:

$$
y[n] - \frac{1}{2}y[n-1] = x[n] \tag{3.7}
$$

where $x[n] = u[n]$ and with a non-initial rest condition of $y[-1] = 1$.
Find the output $y[n]$ of the system.

Solution:
We know that:

$$
X(z) = \mathcal{X}(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1
$$

Taking the unilateral z-transform of (3.7), we get:

$$
\mathcal{Y}(z) - \frac{1}{2}\left\{y[-1] + \mathcal{Y}(z)z^{-1}\right\} = \frac{1}{1-z^{-1}} = \mathcal{X}(z)
$$

Solving for $\mathcal{Y}(z)$,

$$\begin{aligned}
\mathcal{Y}(z) &= \frac{1}{1-\frac{1}{2}z^{-1}}\left\{\frac{1}{2}y[-1]+\frac{1}{1-z^{-1}}\right\} \\
&= \frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}+\frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})} \\
&= \frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{2}{1-z^{-1}} \quad \text{(by partial fraction)} \\
&= \frac{2}{1-z^{-1}}-\frac{1}{2}\frac{1}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad y[n] &= 2u[n]-\frac{1}{2}\left(\frac{1}{2}\right)^{n}u[n] \quad \text{(by inverse unilateral z-transform)} \\
&= \left\{1-\left(\frac{1}{2}\right)^{n+1}\right\}u[n]
\end{aligned}$$

(cf.)

(i) $y[-1]=1$.

(ii) If there is no non-initial condition (i.e. if $y[n]=0, \ \forall\, n<0$), then:

$$\begin{aligned}
& Y(z)-\frac{1}{2}Y(z)z^{-1}=X(z) \\
\to \quad & \left(1-\frac{1}{2}z^{-1}\right)Y(z)=X(z) \\
\to \quad & Y(z)=\frac{1}{1-\frac{1}{2}z^{-1}}\cdot\frac{1}{1-z^{-1}}=\frac{-1}{1-\frac{1}{2}z^{-1}}+\frac{2}{1-z^{-1}} \\
\to \quad & y[n]=\left\{-\left(\frac{1}{2}\right)^{n}+2\right\}u[n]
\end{aligned}$$