Contents

Chapter 4

Sampling of Continuous-Time Signals

4.1 Periodic sampling

Recall: Most of discrete-time signals(i.e. sequences) come from sampling continuous-time signals...

Figure 4.1: Periodic sampling of $x_c(t)$ to yield $x[n] = x_c(nT)$.

T: sampling period(sec)

$\Omega_s \triangleq \frac{2\pi}{T}$: sampling frequency(rad/sec)

Figure 4.2: A C/D converter.

Remarks:

(i) C/D stands for *Continouous to Discrete*.

(ii) Generally better known A/D (*Analog to Digital*) converter is an approximation, since it involes an approximate operation $\ni$: **quantization** etc..

(iii) C/D operation is NOT invertible in general, but by putting some restrictions on $x_c(t)$, such as *bandlimited* and so on, we can completely reconstruct $x_c(t)$ from $x[n]$.

4.2 Analysis of sampling in frequency domain

More detailed representation of C/D conversion is as follows: [1]

Figure 4.3: A detailed representation of C/D converter.

Here, the sampling signal $s(t)$ is a train of impulses, i.e.:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Figure 4.4: The sampling signal $s(t)$: train of impulses.

Therefore, the sampled signal $x_s(t)$ can be expressed as:

$$\begin{aligned} x_s(t) &= x_c(t) \cdot s(t) \\ &= x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &\overset{\text{OR}}{=} \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \end{aligned}$$

: time domain representation

[1]Notice that the **area** of $\delta(t)$ is now converted to the **magnitude** of $\delta[n]$.

We now take the Fourier transform of $x_s(t)$, and by the modulation property of F.T., we get:

$$\begin{aligned} X_s(\Omega) &= \frac{1}{2\pi} X_c(\Omega) * S(\Omega) \\ &= \frac{1}{2\pi} X_c(\Omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s) \qquad (4.1) \\ & \qquad (4.2) \end{aligned}$$

: frequency domain representation

Note:

(i) We use the notation Ω(rad/sec) for the frequency of continuous signals, in order to distinguish it from the discrete frequency ω(rad).

(ii) In above equation, $\Omega_s \triangleq \frac{2\pi}{T}$(rad/sec) is the sampling frequency of the C/D converter.

Graphical Interpretation:

Let a "bandlimited" continuous-time signal $x_c(t)$ have the following spectrum:

Figure 4.5: The Fourier transform of a bandlimited signal $x_c(t)$.

Then, the spectrum (i.e. Fourier transform) $X_s(\Omega)$ of the sampled signal $x_s(t)$ is in the following form, which is the replica of scaled and shifted $X_c(\Omega)$:

1. Case#1: when $\Omega_M \leq \Omega_s - \Omega_M$ (i.e. $\Omega_s \geq 2\Omega_M$)

Figure 4.6: The Fourier transform $X_s(\Omega) : \Omega_s \geq 2\Omega_M$.

2. Case#2: when $\Omega_M > \Omega_s - \Omega_M$(i.e. $\Omega_s < 2\Omega_M$)

Figure 4.7: The Fourier transform $X_s(\Omega) : \Omega_s < 2\Omega_M$.

   In this case, the spectrum $X_s(\Omega)$ is completely different from that of $X_c(\Omega)$, and it is referred to as **"ALIASING"**.

$\Longrightarrow$ Only for the first case, i.e. when $\Omega_s \geq 2\Omega_M$, we can recover (reconstruct) $x_c(t)$ from the sampled signal $x_s(t)$ via a low pass filter of which the transfer function $H(\Omega)$ is as follows:

Figure 4.8: The transfer function $H(\Omega)$ of the reconstruction filter.

where the cutoff frequency Ω_c must satisfy $\Omega_M < \Omega_c < \Omega_s - \Omega_M$, and we typically choose $\Omega_c = \Omega_s/2$.

Theorem 4.1 NYQUIST SAMPLING THEOREM:

Let $x_c(t)$ be a bandlimited signal, i.e.

$$X_c(\Omega) = 0, \qquad |\Omega| > \Omega_M$$

Then, $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT), \ -\infty < n < \infty$, **if:**

$$\Omega_s \geq 2\,\Omega_M$$

where $\Omega_s = \frac{2\pi}{T}$ is the sampling frequency.

(cf.) We call Ω_M and $2\Omega_M$ the Nyquist frequency and the Nyquist rate of $x_c(t)$ respectively.

DTFT of $\mathbf{x[n] = x_c(nT)}$: in terms of $X_c(\Omega) = \mathcal{F}\{x_c(t)\}$

Since the sampled signal $x_s(t)$ can be represented as follows:

$$\begin{aligned} x_s(t) &= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT) \end{aligned}$$

By taking the Fourier transform of both sides, we have:

$$\begin{aligned} \mathcal{F}\{x_s(t)\} = X_s(\Omega) &\equiv \sum_{n=-\infty}^{\infty} x[n]1 \cdot e^{-j\Omega nT} \\ &\triangleq X\left(e^{j\Omega T}\right) \\ &= F\{x[n]\}_{\omega=\Omega T} \end{aligned}$$

which renders the following relationship: [2]

$$X\left(e^{j\Omega T}\right) = X_s(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(\Omega - k\Omega_s)$$

where $\Omega_s = \frac{2\pi}{T}$(rad/sec).

[2]Here we quote (4.1), the F.T. of the sampled signal $x_s(t)$.

By applying the change of variable as:

$$\omega = \Omega T$$

we can obtain the following relationship, which represents tha DTFT of the sampled sequence $x[n]$ in terms of the F.T. of the original continuous signal $x_c(t)$:

$$\begin{aligned} X\left(e^{j\omega}\right) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right) \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi k}{T}\right) \end{aligned}$$

: frequency scaled version of $X_s(\Omega)$ via $\omega = \Omega T$

(e.g.)

If the continuous frequency is $\Omega = \Omega_s$(rad/sec), then the corresponding discrete frequency becomes $\omega = \Omega_s T = \frac{2\pi}{T}T = 2\pi$(rad) .

: normalization of frequency axis

Remark: Signal representation in the sampling process:

(1) Time domain:

Figure 4.9: Continuous and sampled signals in time domain.

(2) Frequency domain:

Figure 4.10: Corresponding spectra in frequency domain.

4.3 Reconstruction of a bandlimited signal: Interpolation

Remark:

From the sampling theorem, as long as a sequence $x[n]$ is sampled from $x_c(t)$ satisfying the Nyquist criterion, the original continuous signal $x_c(t)$ can be recovered by an ideal LPF:

Figure 4.11: The block diagram of interpolation: D/C converter.

where the transfer function $H_r(\Omega) = \mathcal{F}(h_r(t))$ of the reconstruction (ideal lowpass) filter is as follows:

Figure 4.12: The transfer function of the ideal LPF.

Here, $\Omega_s/2 = \frac{\pi}{T}$ is called the *folding frequency*, and the cutoff frequency of the reconstruction filter should meet: [3]

$$\Omega_M \leq \Omega_s < \Omega_s - \Omega_M$$

and WLOG [4] , we usually let $\Omega_c = \frac{\pi}{T} = \frac{\Omega_s}{2}$(rad/sec)

[3] Ω_M represents the maximum frequency in $x_c(t)$.

[4] WLOG: without loss of generality

Analysis of D/C in time domain:

Recall that the sampled signal can be represented as:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

: sequence to weighted impulse train

and, since the ideal LPF is an LTI system with impulse response of $h_r(t) \triangleq L[\delta(t)]$, we have:

$$\begin{aligned} x_r(t) &= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sinc}\left(\frac{t - nT}{T}\right) \end{aligned}$$

where [5] [6]

$$\begin{aligned} h_r(t) &= \mathcal{F}^{-1}\{H_r(\Omega)\} \\ &= \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \\ &\triangleq \text{sinc}\left(\frac{t}{T}\right) \end{aligned}$$

Figure 4.13: The impulse response of the ideal (reconstructing) LPF.

$\Longrightarrow$ **We expect $x_r(t) = x_c(t)$ if the sampling period T satisfies the Nyquist criterion.**

[5]recall from the Signals and Systems class...

[6]Note that:

$$h_r(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Interpolation:

Figure 4.14: The interpolation process.

Note: $x_r(mT) = x_c(mT)$

$$\begin{aligned} x_r(mT) &= \sum_{n=-\infty}^{\infty} x_c(nT)h_r(mT - nT) \\ &= \sum_{n=-\infty}^{\infty} x_c(nT)h_r\left((m-n)T\right) \\ &= x_c(mT) \end{aligned}$$

which means that the original continuous signal $x_c(t)$ and the reconstructed signal $x_r(t)$ **exactly match** at least at the time instances of integer multiple of the sampling period T.
In the above deriavtion, we have used the fact:

$$h_r(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

I/O relationship of D/C in frequency domain:

Figure 4.15: The D/C conversion.

Recall the interpolation(D/C) formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

By taking the Fourier transform, we get:

$$\begin{aligned}
\mathcal{F}[x_r(t)] \triangleq X_r(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]\mathcal{F}[h_r(t-nT)] \\
&= \sum_{-\infty}^{\infty} x[n]H_r(\Omega)e^{-j\Omega nT} \\
&= H_r(\Omega)\sum_{-\infty}^{\infty} x[n]e^{-j\Omega nT} \\
&= H_r(\Omega)F\{x[n]\}_{\omega=\Omega T} \\
&= H_r(\Omega)X\left(e^{j\Omega T}\right)
\end{aligned}$$

i.e.: [7]

$$\begin{aligned}
X_r(\Omega) &= H_r(\Omega)\cdot X\left(e^{j\Omega T}\right) \\
&\overset{\text{recall}}{=} H_r(\Omega)\cdot X_s(\Omega)
\end{aligned}$$

$\Longrightarrow$ **We expect $X_r(\Omega) = X_c(\Omega)$ if the reconstruction filter $H_r(\Omega)$ is an ideal LPF.**

[7] Notice that $X_r(\Omega)$ and $H_r(\Omega)$ represent the *continuous* signal and system respectively, whereas $X\left(e^{j\Omega T}\right)$ represents the *discrete* signal.

4.4 Discrete-time processing of continuous signals

4.4.1 Effective (equivalent) continuous system

General block diagram: [8]

where $Y_r(\Omega) = H_{\text{eff}}(\Omega) \cdot X_c(\Omega)$

Figure 4.16: A DSP system and its equivalent continuous system.

(cf.)
We assume that C/D and D/C converters have the same sampling period (T).

Objective: Find $H_{\text{eff}}(\Omega)$ in terms of $H(e^{j\omega})$.

Let

$$X_c(\Omega) = \mathcal{F}\{x_c(t)\}, \quad Y_r(\Omega) = \mathcal{F}\{y_r(t)\}$$

$$X\left(e^{j\omega}\right) = F\{x[n]\}, \quad Y\left(e^{j\omega}\right) = F\{y[n]\}$$

[8] This is the same as the typical DSP system discussed in Chapter 2.

First, consider the analog parts((1) and (3) in above figure):

The input/output relations of the C/D and the D/C converters can be represented in the frequency domain respectively as follows:

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - k\cdot\frac{2\pi}{T}\right) \tag{4.3}$$

and

$$Y_r(\Omega) = H_r(\Omega)\cdot Y\left(e^{j\Omega T}\right) = \begin{cases} T\cdot Y\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases} \tag{4.4}$$

where we assumed $H_r(\Omega)$ is an ideal LPF with gain of T as follows:

Figure 4.17: $H_r(\Omega)$ as an ideal LPF.

Now, consider the discrete part((2) in above figure):

Since the discrete system is an LTI system, we have:

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)\cdot X\left(e^{j\omega}\right) \tag{4.5}$$

where $H\left(e^{j\omega}\right)$ is the frequency response of the discrete system.

Assuming:

(i) $X_c(\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T}$ (bandlimited)

(ii) $H_r(\Omega)$ is an ideal LPF with gain of T (reconstruction filter)

(iii) T satisfies the Nyquist criterion, i.e. $T < \frac{\pi}{\Omega_M}$(sec).

we have from (4.3), (4.4), and (4.5):

$$\begin{aligned}
Y_r(\Omega) &= H_r(\Omega) \cdot Y\left(e^{j\Omega T}\right) \\
&= H_r(\Omega) \cdot H\left(e^{j\Omega T}\right) \cdot X\left(e^{j\Omega T}\right) \\
&= H_r(\Omega) \cdot H\left(e^{j\Omega T}\right) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\Omega - k\frac{2\pi}{T}\right) \\
&= T \cdot H\left(e^{j\Omega T}\right) \cdot \frac{1}{T} X_c(\Omega) \\
&= H\left(e^{j\Omega T}\right) \cdot X_c(\Omega) \qquad \text{where } |\Omega| < \tfrac{\pi}{T}
\end{aligned}$$

from which the following relation must hold:

$$Y_r(\Omega) = H\left(e^{j\Omega T}\right) \cdot X_c(\Omega) \equiv H_{\text{eff}}(\Omega) \cdot X_c(\Omega)$$

where $|\Omega| < \frac{\pi}{T}$.

Therefore, an equivalent continuous-time system for the entire DSP system can be described as follows: [9]

Figure 4.18: An equivalent continuous LTI system.

where

$$H_{\text{eff}}(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases}$$

[9]Be reminded that $H\left(e^{j\Omega T}\right)$ is periodic.

Example 4.1

Consider a discrete LTI system with the frequency response $H\left(e^{j\omega}\right)$ of the following form:

Figure 4.19: The frequency response of a discrete LTI system.

Then, since

$$H_{\text{eff}}(\Omega) = H\left(e^{j\Omega T}\right), \qquad |\Omega| < \frac{\pi}{T}$$

we have the equivalent continuous LTI system with the following transfer function:

$$H_{\text{eff}}(\Omega) = H\left(e^{j\Omega T}\right) = \begin{cases} 1. & |\Omega T| \leq \omega_c \quad (\text{or } \ |\Omega| \leq \frac{\omega_c}{T}) \\ 0, & \text{elsewhere} \end{cases}$$

Figure 4.20: The transfer function of the equivalent continuous LTI system.

And the following two systems are equivalent in operation:

Figure 4.21: The equivalent DSP and continuous LTI systems.

Illustration:

Suppose $x_c(t)$ is a bandlimited signal with $X_c(\Omega)$ of:

Figure 4.22: The F.T of a bandlimited continous signal $x_c(t)$

and let T be chosen $\ni$: $\Omega_N > \frac{\omega_c}{T} = \Omega_c$ where ω_c is given. [10]

(1) Continuous system:

$$\text{where } \Omega_c \triangleq \frac{\omega_c}{T} \text{ and } T \cdot \Omega_N > \omega_c \text{ by assumption}$$

Figure 4.23: The output spectrum $Y_r(\Omega)$ through continuous system.

[10]This determines the overall system's characteristics, i.e. some portions of the input frequencies are cut off.

(2) Discrete(DSP) system:

Figure 4.24: The output spectrum $Y_r(\Omega)$ through DSP system.

Notice that we have the same result !!!

NOTE:
The cut-off frequency of the effective continuous system depends both on ω_c and T (sampling period) via:

$$\Omega_c = \frac{\omega_c}{T}$$

$\Longrightarrow$ **With a given (fixed) discrete system w/ specific ω_c, we can implement an equivalent continuous system w/ a varying cut-off frequency (Ω_c) by adjusting the sampling period T, i.e. :**

$$\mathbf{\Omega_c \propto \frac{1}{T}}$$

(e.g.) Choose $T \ni: T \cdot \Omega_N < \omega_c$ in the previous example, then the equivalent continuous system becomes:

Figure 4.25: The effective conti-system with different T.

and in this case, we expect:

$$y_r(t) = x_c(t)$$

Assignment: Problem 3.11

4.4.2 Impulse invariant systems

We are given an analog system with $H_c(\Omega)$, and want to design an equivalent discrete system: [11]

Figure 4.26: The concept of the impulse invariant systems.

Objective: Find $h[n]$ in terms of sampled version of $h_c(t)$.

Recall that

$$H_c(\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases}$$

Let $\omega = \Omega T$, then we have:

$$H\left(e^{j\omega}\right) = H_c\left(\frac{\omega}{T}\right), \quad |\omega| < \pi \quad (\text{period} = 2\pi) \tag{4.6}$$

[11] This is converse to the concept discussed in the previous section, i.e. the effective continuous system.

Now, let the sampled (T) version of the impulse response $h_c(t)$ be $h_d[n]$, i.e. $h_d[n] = h_c(nT)$, then:

$$H_d\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)$$

or

$$H_d\left(e^{j\omega}\right) = \frac{1}{T}H_c\left(\frac{\omega}{T}\right), \qquad |\omega| < \pi \tag{4.7}$$

Comparing (4.6) and (4.7), we get:

$$H\left(e^{j\omega}\right) = T \cdot H_d\left(e^{j\omega}\right)$$

and by taking the inverse DTFT, we obtain:

$$\xrightarrow{F^{-1}} \quad h[n]\,(= T \cdot h_d[n]) = T \cdot h_c(nT)$$

$\Longrightarrow$ **The impulse response $h[n]$ of the equivalent discrete system is a *scaled*, *sampled* version of the impulse response $h_c(t)$ of the continuous system.**

$\Longrightarrow$ **$h[n]$ is called the impulse invariant version of the continuous systrem!!!**

4.5 Continuous-time processing of discrete-time signals

Following discussion are not typically used to implement discrete systems, but its theoretical analysis provides useful interpretations and insights for discrete systems.....

General block diagram:

$$\text{where } Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) \cdot X\left(e^{j\omega}\right)$$

Figure 4.27: A conti-system and its equivalent discrete system.

We assume that:

(i) $X_c(\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T}$ (bandlimited) [12]

(ii) $H_r(\Omega)$ is an ideal LPF with gain of T, and $\Omega_c = \frac{\pi}{T}$.

[12] Therefore, $Y_c(\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T}$ as well.

Then, we have the following input/output relationships for each part of the overall continuous system:

(1) D/C converter: [13]

$$\begin{cases} x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\text{sinc}\left(\frac{t-nT}{T}\right) & : \text{time} \\ X_c(\Omega) = T \cdot X\left(e^{j\Omega T}\right), \quad |\Omega| < \frac{\pi}{T} & : \text{frequency} \end{cases} \tag{4.8}$$

(2) C/D converter: [14]

$$\begin{cases} y_c(t) = \sum_{n=-\infty}^{\infty} y[n]\text{sinc}\left(\frac{t-nT}{T}\right) & : \text{time} \\ Y\left(e^{j\omega}\right) = \frac{1}{T} Y_c\left(\frac{\omega}{T}\right), \quad |\omega| < \pi & : \text{frequency} \end{cases} \tag{4.9}$$

(3) Conti-system:

$$\begin{cases} y_c(t) = h_c(t) * x_c(t) & : \text{time} \\ Y_c(\Omega) = H_c(\Omega) \cdot X_c(\Omega) & : \text{frequency} \end{cases} \tag{4.10}$$

Inserting (4.8) and (4.10) into (4.9), we get:

$$\begin{aligned} Y\left(e^{j\omega}\right) &= \frac{1}{T} Y_c\left(\frac{\omega}{T}\right) \\ &= \frac{1}{T} H_c\left(\frac{\omega}{T}\right) X_c\left(\frac{\omega}{T}\right) \\ &= \frac{1}{T} H_c\left(\frac{\omega}{T}\right) T X\left(e^{j\omega}\right) \\ &= H_c\left(\frac{\omega}{T}\right) X\left(e^{j\omega}\right), \qquad |\omega| < \pi \end{aligned}$$

: equivalent I/O relationship for the discrete system

[13] Note that $X_c(\Omega) = H_r(\Omega) \cdot X\left(e^{j\Omega T}\right)$, where $H_r(\Omega) = 0$ for $|\Omega| > \frac{\pi}{T}$.

[14] In this case, $y[n] = y_c(nT)$ and $Y\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$ which is periodic in ω w/ period 2π.

Therefore, we have:

$$H\left(e^{j\omega}\right) = H_c\left(\frac{\omega}{T}\right), \quad |\omega| < \pi \quad \text{periodic } (2\pi)$$

OR

$$H_c(\Omega) = H\left(e^{j\Omega T}\right), \quad |\Omega| < \frac{\pi}{T}$$

Example 4.2

Consider a discrete system w/ frequency response of:

$$H\left(e^{j\omega}\right) = e^{-j\omega\Delta}, \qquad |\omega| < \pi$$

Then, the impulse response is:

$$h[n] = \delta[n - \Delta]$$

and the i/o of the system can be represented as:

$$y[n] = x[n - \Delta]$$

which is the *ideal delay.*

Figure 4.28: A DLTI system(e.g. ideal delay).

If Δ is an integer, $y[n]$ is just a shifted version of $x[n]$, but if Δ is not an integer, how do we interprete this? [15]

Solution:

In this case, the equivalent continuous system becomes:

$$H_c(\Omega) = H\left(e^{j\Omega T}\right) = e^{-j\Omega\delta T}, \quad |\Omega| < \frac{\pi}{T}$$

[15]Notice that $x[n - \Delta]$ does not have any formal meaning by itself when δ is not an integer.

Corresponding (continuous) impulse response and the output signals are respectively:

$$h_c(t) = \delta(t - \Delta T)$$

$$y_c(t) = x_c(t - \Delta T)$$

and if we take samples of $y_c(t)$ with sampling period T(i.e. C/D conversion), we obtain $y[n]$.

(e.g.) If $\delta = \frac{1}{2}$, then:

$$y_c(t) = x_c(t - \frac{T}{2}) \quad \xrightarrow{(T)} \quad y[n] \text{ (: C/D conversion)}$$

Figure 4.29: $y[n]$ sampled from $y_c(t)$.

Therefore, we can interprete $y[n] = x[n - \Delta]$, where Δ is not an integer, as a sampled sequence of $x_c(t - \Delta T) = y_c(t)$!!!

(Although $y[n] = x[n - \Delta]$ by itself does not have any meaning.....)

4.6 Changing the sampling rate using discrete-time processing

Objective: We want to change the sampling rate from T_1 to T_2

1. $T_2 = M \cdot T_1$ where M is an integer.
2. $T_2 = 1/L \cdot T_1$ where L is an integer.
3. $T_2 = \alpha \cdot T_1$ where α is a real number.

Ordinary way: [16]

Figure 4.30: Changing sampling period from T_1 to T_2.

Question: How do we get $x_2[n]$ directly from $x_1[n]$?

(How is $X_2\left(e^{j\omega}\right)$ related to $X_1\left(e^{j\omega}\right)$ in frequency domain?)

[16] Note that in this way, we cannot accomplish exact change of sampling rate, since C/D and D/C are imperfect operations in practice.

4.6.1 Reduction by an integer factor (downsampling or deciamtion)

Figure 4.31: Downsampling: Decimation by an integer factor.

(cf.) This system is called the *(sampling rate) "compressor"* : reampling.

Remark:

Suppose $X_c(\Omega) = 0, \quad |\Omega| > \Omega_N$, then $x_c(t)$ can be completely recovered from $x_d[n]$ **IF:**

$$\frac{2\pi}{T_2} = \frac{2\pi}{M \cdot T_1} > \Omega_N$$

$$\text{i.e. } \frac{\pi}{T_1} > M \cdot \Omega_N$$

$$\Rightarrow \quad \frac{2\pi}{T_1} > M \cdot (2\Omega_N)$$

Therefore, the original sampling rate must be at least M times the Nyquist rate!!!

Frequency domain relation: $\{X(e^{j\omega})$ vs. $X_d(e^{j\omega})\}$

$$X\left(e^{j\omega}\right) = \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_1} - \frac{2\pi k}{T_1}\right)$$

$$\begin{aligned} X_d\left(e^{j\omega}\right) &= \frac{1}{T_2} \sum_{r=-\infty}^{\infty} X_c\left(\frac{\omega}{T_2} - \frac{2\pi r}{T_2}\right) \\ &= \frac{1}{MT_1} \sum_{r=-\infty}^{\infty} X_c\left(\frac{\omega}{MT_1} - \frac{2\pi r}{MT_1}\right) \end{aligned}$$

Let $r = i + k \cdot M$, where $0 \le i \le M-1$, and $-\infty < k < \infty$, then $-\infty < r < \infty$.

(cf.)

Figure 4.32: Change of integer variable: $r = i + kM$.

Therefore, we have:

$$\begin{aligned} X_d\left(e^{j\omega}\right) &= \frac{1}{M} \sum_{i=0}^{M-1} \left\{ \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{MT_1} - \frac{2\pi k}{T_1} - \frac{2\pi i}{MT_1}\right) \right\} \\ &= \frac{1}{M} \sum_{i=0}^{M-1} \left\{ \frac{1}{T_1} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi i}{MT_1} - \frac{2\pi k}{T_1}\right) \right\} \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(e^{j\frac{\omega - 2\pi i}{M}}\right) \end{aligned}$$

: M **copies of** $\frac{1}{M}X(e^{j\omega})$ **frequency scaled by** M **and shifted by** $2\pi i$**,** $(i = 0, 1, 2, \ldots, M-1)$

Example 4.3

Suppose $x_c(t)$ is bandlimited by $X_c(\Omega) =, \quad |\Omega| > \Omega_N$, and let the sampling period T be chosen such that: [17]

$$\frac{2\pi}{T} = 4 \cdot \Omega_N \qquad (\text{i.e. } T = \tfrac{\pi}{2\Omega_N})$$

Figure 4.33: A bandlimited $X_c(\Omega)$ w/ max imum frequency of Ω_N.

Figure 4.34: Downsampler by M.

(1) Case of $M = 2$:

The original sampled sequence $x[n]$ has the followng spectrum:

$$X\left(e^{j\omega}\right) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

$$\text{where } \omega_N = \Omega_N T = \Omega_N \cdot \frac{\pi}{2\Omega_N} = \frac{\pi}{2}$$

Figure 4.35: Spectrum $X\left(e^{j\omega}\right)$.

[17]Notice that the sampling rate is twice the Nyuquist rate, i.e. $\frac{2\pi}{T} = 2 \cdot (2\Omega_N)$.

After downsampling($M=2$), the decimated spectrum would be: [18]

$$
\begin{aligned}
X_d\left(e^{j\omega}\right) &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega-2\pi i}{M}}\right) \\
&= \frac{1}{2}\sum_{i=0}^{1} X\left(e^{j\frac{\omega-2\pi i}{2}}\right) \\
&= \begin{cases} \frac{1}{2}X\left(e^{j\frac{\omega}{2}}\right), & i=0 \\ \\ \frac{1}{2}X\left(e^{j\frac{\omega-2\pi}{2}}\right), & i=1 \end{cases}
\end{aligned}
$$

Figure 4.36: Spectrum $X_d\left(e^{j\omega}\right)$.

Remark:

Notice that the aliasing does not occurr, since the original sampling rate satisfies: $\frac{2\pi}{T} \geq M \cdot (2\Omega_N) = 4\Omega_N$.

General condition to avoid aliasing by downsampling by M:

$$\Omega_s = \frac{2\pi}{T} \geq M \cdot (2\Omega_N)$$

$$\longrightarrow \quad \frac{2\pi}{T} \geq M \cdot 2 \cdot \frac{\omega_N}{T}$$

$$\longrightarrow \quad \omega_N \leq \frac{\pi}{M}$$

i.e.: The maximum (highest) frequency ω_N in $x[n]$ should be less than $\frac{\pi}{M}$(rad).

[18] Note that $\omega^{'} = M\omega = 2\omega$ in this case.

(2) Case of $M = 3$: [19]

The original sampled sequence $x[n]$ is the same as before:

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)$$

$$\text{where } \ \omega_N = \Omega_N T = \Omega_N \cdot \frac{\pi}{2\Omega_N} = \frac{\pi}{2}$$

Figure 4.37: Spectrum $X\left(e^{j\omega}\right)$.

After downsampling($M = 3$), the decimated spectrum would be: [20]

$$\begin{aligned} X_d\left(e^{j\omega}\right) &= \frac{1}{M}\sum_{i=0}^{M-1} X\left(e^{j\frac{\omega - 2\pi i}{M}}\right) \\ &= \frac{1}{3}\sum_{i=0}^{2} X\left(e^{j\frac{\omega - 2\pi i}{3}}\right) \\ &= \begin{cases} \frac{1}{3}X\left(e^{j\frac{\omega}{3}}\right), & i = 0 \\ \frac{1}{3}X\left(e^{j\frac{\omega - 2\pi}{3}}\right), & i = 1 \\ \frac{1}{3}X\left(e^{j\frac{\omega - 4\pi}{3}}\right), & i = 2 \end{cases} \end{aligned}$$

Figure 4.38: Spectrum $X_d\left(e^{j\omega}\right)$.

Remark:

Notice that the aliasing does really occurrs!!!

[19] In this case, aliasing will occur since $\omega_N = \frac{\pi}{2} > \frac{\pi}{3} = \frac{\pi}{M}$.

[20] Note that $\omega^{'} = M\omega = 3\omega$ in this case.

Remark:
To avoid the aliasing phenomenon by downsampling, we must sacrifice some portions of signal bandwidth by low pass filtering:

Since the highest frequency for $x[n]$ is $\omega_N = \frac{\pi}{M}$(rad), in order to avoid aliasing by downsampling (M), we first pass $x[n]$ through a LPF with the following frequency response $H_d\left(e^{j\omega}\right)$:

Figure 4.39: The pre-filter $H_d\left(e^{j\omega}\right)$: period=$2\pi$.

Figure 4.40: The block diagram of "decimator".

Illustration:

Figure 4.41: The spectra of signals during decimation process.

Note:
Notice that $\tilde{x}_d[n]$ corresponds to the sampled version of $\tilde{x}_c(t)$, which is the output of $x_c(t)$ throught a LPF w/ following transfer function, where the cutoff frequency is $\Omega_M = \frac{\pi}{T \cdot M}$ (rad/sec):

Figure 4.42: The continuous counterpart of $\tilde{x}_d[n]$.

4.6.2 Increasing by an integer factor (upsampling or interpolation)

Figure 4.43: Upsampling: Interpolation by an integer factor.

(cf.) This system is called the *(sampling rate) "expander"* : i.e., increasing the # of points(samples) by L.

Illustration:
Let $x_c(t)$ be as before, and assume that the sampling rate has been taken $\ni$: $\frac{2\pi}{T_1} = 2\Omega_N$, i.e. $\Omega_N = \frac{\pi}{T_1}$: which means that the sampling period T_1 is chosen *just to avoid aliasing*!
Suppose $L = 2$, then we expect that the desired $x_i[n]$ should have the following spectrum, where:

$$X_i\left(e^{j\omega}\right) = \frac{1}{T_2} \sum_{k=-\infty}^{\infty} X_c\left(\frac{\omega}{T_2} - \frac{2\pi k}{T_2}\right)$$

Figure 4.44: The desired interpolated spectrum $X_i\left(e^{j\omega}\right)$.

(1) Analysis (frequency domain) :

Let's define:

$$
\begin{aligned}
x_e[n] &\triangleq \begin{cases} x\left[\frac{n}{L}\right], & n = k \cdot L \\ 0, & n \neq k \cdot L \end{cases} \\
&= \sum_{k=-\infty}^{\infty} x[k]\delta[n - k \cdot L]
\end{aligned}
$$

(cf.) Note that time axis n is scaled bt $\frac{1}{L}$ for expansion.

Figure 4.45: Example of expansion for $L = 2$.

Taking the DTFT of the expanded sequence $x_e[n]$, we obtain:

$$
\begin{aligned}
X_e\left(e^{j\omega}\right) = F\left\{x_e[n]\right\} &= \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k L} \\
&\triangleq X\left(e^{j\omega L}\right)
\end{aligned}
$$

: frequency scaled version of $X\left(e^{j\omega}\right)$
(compressed by L)

Figure 4.46: The spectra for the process of upsampling when $L = 2$.

(2) Analysis (time domain) : interpolation

This is for partial verification of $x_i[n] = x_c(T_2 n)$ for $n = kL$:

Figure 4.47: Example of sequence for upsampling when $L = 2$.

Notice that the impulse response of the discrete LPF is as follows:

$$\begin{aligned} h_i[n] = F^{-1}\left\{H_i\left(e^{j\omega}\right)\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_i\left(e^{j\omega}\right) e^{j\omega n} d\omega \\ &\quad \vdots \ \text{(assignment)} \\ &= \text{sinc}\left[\frac{n}{L}\right] \end{aligned}$$

Note: Check that $h_i[n]$ has the following characteristics:

$$h_i[n] = \frac{\sin\left(\frac{\pi n}{L}\right)}{\frac{\pi n}{L}} = \begin{cases} 1, & n = 0 \\ 0, & n = k \cdot L \end{cases}$$

Figure 4.48: The impulse response $h_i[n]$ when $L = 2$.

Since

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$$

we have:

$$\begin{aligned} x_i[n] &= \sum_{k=-\infty}^{\infty} x[k]\cdot h_i[n-kL] \\ &= \sum_{k=-\infty}^{\infty} x[k]\cdot \text{sinc}\left[\frac{n-kL}{L}\right] \\ &\quad : \text{sinc interpolation} \\ &\underline{\text{or}} \sum_{k=-\infty}^{\infty} x[k]\cdot \text{sinc}\left[k-\frac{n}{L}\right] \end{aligned}$$

Remark:

Note that when $n = kL$ (i.e. $k = \frac{n}{L}$) we have:

$$x_i[n] = x\left[\frac{n}{L}\right]$$

as we eapected!!!

(cf.) In fact,

$$x_i[n] = x_c\left(\frac{T_1}{L}n\right) = x_c(T_2\cdot n)$$

from the analysis in frequency domain in (1).

Practical consideration (Approximation)

Since ideal LPF cannot be implemented in practice, we replace $h_i[n]$ with $h_{\text{lin}}[n]$ defined as:

$$h_{\text{lin}}[n] \triangleq \begin{cases} 1 - \frac{|n|}{L}, & |n| < L \\ 0, & \text{elsewhere} \end{cases}$$

(e.g.) $L = 3$:

Figure 4.49: The linear approximation of the ideal LPF's impulse response $h_i[n]$.

Then,

$$x_{\text{lin}}[n] = \sum_{k=-\infty}^{\infty} x[k] h_{\text{lin}}[n - kL]$$

: lineal interpolation

$\Rightarrow$ Some errors must naturally occur by using $h_{\text{lin}}[n]$ in place of $h_i[n]$.

Note:

(i) $h_{\text{lin}}[n]$ has the same characteristics as $h_i[n]$ such that:

$$h_{\text{lin}}[n] = \begin{cases} 1, & n = 0 \\ 0, & n = k \cdot L \qquad (\text{since } |n| > L) \end{cases}$$

(ii) The DTFT of $h_{\text{lin}}[n]$ is as follows: (**proof:** assignment)

$$H_{\text{lin}}\left(e^{j\omega}\right) = \frac{1}{L} \left\{ \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right\}^2$$

Figure 4.50: The linear approximation $H_{\text{lin}}\left(e^{j\omega}\right)$ and $H_i\left(e^{j\omega}\right)$.

4.6.3 Changing by non-integer factor (upsampling and downsampling)

By combining the *decimator* and the *interpolator*, we can achieve any desired sampling rates, i.e.

where $\frac{M}{L}$ could be any rational real number

Figure 4.51: Combination of decimator and interpolator.

Remark:

Since $H_i\left(e^{j\omega}\right)$ and $H_d\left(e^{j\omega}\right)$ are in cascade, we can merge (combine) two LPF's into one, i.e.:

Figure 4.52: Combination of decimator and interpolator w/ single LPF.

(i) $M > L$: downsampling

(ii) $M < L$: upsampling

4.7 Practical considerations

Practical restrictions on C/D and D/C:

(1) $x_c(t)$ is not precisly bandlimited.

(2) Ideal (analog) filters cannot be realized.

(3) C/D and D/C converters can only be approximated due to limitations on digital hardwares (i.e. quantization) : replaced by A/D and D/A converters.

4.7.1 Prefiltering to avoid aliasing

Necessity: (two-fold)

(i) $x_c(t)$ is not usually bandlimited, i.e. $\Omega_N \gg \frac{\Omega_s}{2}$, where Ω_N is the maximum frequency of $x_c(t)$ and Ω_s is the sampling frequency that is fixed by the given hardware.

(ii) The existence of wideband additive noise, even though $x_c(t)$ is bandlimited.

$\Longrightarrow$ In these situations, we must use a prefilter before C/D conversion to avoid aliasing phenomenon forcing the frequencies of the input signal less than one-half ($\frac{1}{2}$) of the sampling frequency.

$\Longrightarrow$ called **anti-aliasing filter**: (*ideal*)

$$H_{aa}(\Omega) = \begin{cases} 1, & |\Omega| \le \Omega_c \le \frac{\pi}{T} = \frac{\Omega_s}{2} \\ 0, & |\Omega| > \Omega_c \end{cases}$$

Remark: In practice, this anti-aliasing filter should also be approximated.

Figure 4.53: An anti-aliasing filter.

(cf.) Notice that $H_2(\Omega)$ can further reduce the effect of the noise compared to $H_1(\Omega)$ $\longrightarrow$ higher SNR!

Example 4.4

Speech signal processing:

Figure 4.54: An anti-aliasing filter for audible signals.

Typically, we have:

$$x_c(t) \quad : \quad 4 \sim 20\text{KHz}$$

$$x_a(t) \quad : \quad 3 \sim 4\text{KHz}$$

and $x_a(t)$ is usually sufficient for intelligibility.

Advantage: in addition to anti-aliasing effect

We can reduce the sampling rate from T_2 to T_1 where $T_2 \ll T_1$

$\Longrightarrow$ we can reduce the number of samples (or data)

$\Longrightarrow$ we can speed up the processing time

$\Longrightarrow$ we can utilize less expensive hardwares

where

$$T_2 < \frac{1}{2 \times 2 \times 10^4}(\text{sec})$$

$$T_1 < \frac{1}{2 \times 4 \times 10^3}(\text{sec})$$

and

$$T_2 \ll T_1$$

Block diagram:

Figure 4.55: A DSP system including anti-aliasing filter.

where

$$H_{eff}(\Omega) = \begin{cases} H_{aa}(\Omega) \cdot H\left(e^{j\Omega T}\right), & |\Omega| \le \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

This is because the C/D, DLTI, and D/C parts are equivalent to a continuous system $H_e(\Omega) \ni$:

$$H_e(\Omega) = H\left(e^{j\Omega T}\right), \quad |\Omega| \le \frac{\pi}{T}$$

(cf.) Therefore, $H_{aa}(\Omega)$ should be considered as another design factor for the overall system.

4.7.2 Analog to digital (A/D) conversion

We must represent each sample of $x[n]$ with finite precision, since we only have limited number of bits to be used for expressing $x[n]$.

$\Longrightarrow$ **quantization**

Block diagram: (concept)

Figure 4.56: Process of A/D conversion.

Illustration:

Figure 4.57: The output of the sample and hold circuit.

$\Longrightarrow$ For every T seconds, each sample is converted into a binary code representing quantized amplitude. (using a clock of period T)

$\Longrightarrow$ Since A/D conversion is not instantaneous, each sample must be held constant until the next sample comes along. (necessaty of holding)

$\Longrightarrow$ A/D conversion for each sample must be completed within T seconds. (speed limit)

(cf.) As T becomes smaller, we need higher speed A/D converting hardware which will increase the cost!!!

(1) Analysis of sample and hold: (*how to implement*)

Figure 4.58: The sample and hold system.

where

$$h_o(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{elsewhere} \end{cases}$$

Figure 4.59: The impulse response $h_o(t)$ of the holding system.

$$\begin{aligned} x_o(t) &= x_s(t) * h_o(t) \\ &= \left\{ \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \right\} * h_o(t) \\ &= \sum_{n=-\infty}^{\infty} x[n] \left\{ \delta(t - nT) * h_o(t) \right\} \\ &= \sum_{n=-\infty}^{\infty} x[n] h_o(t - nT) \end{aligned}$$

: well matched with $x_o(t)$ in illustration

(2) Mathematical structure of A/D conversion:

Figure 4.60: Detailed procedure of A/D conversion.

(i) **Quantizer:**

Transforms (or maps) the input sample $x[n]$ into one of a *finite set* of prescribed values.

$$\hat{x}[n] = \mathrm{Q}\left(x[n]\right): \quad : \text{ called } \textit{quantized sample}$$

$\Longrightarrow$ Each sample value $x[n]$ should be rounded to the nearest quantization level.

(cf.) We only consider uniformly spaced quatization level.

(ii) **Coder:**

Represents each quantized sample in a *binary codeword.*

Example 4.5

A/D conversion with 3bit machine using *2's complement code* and *binary offset code*, where

$$\begin{cases} \text{number of quantization level} = 8 \\ (B+1)\text{bit coder with } B = 2 \quad (\text{since } 2^3 = 8) \end{cases}$$

Figure 4.61: A/D conversion with 3-bit machine.

Notes:

(i) Note that the MSB represents the *sign* of the amplitude in both of the 2's complement code and the binary offset code.

(ii) X_m is called the *full scale level* of the A/D converter.

Remarks:

(a) Usually the number of quantization level is a power of 2 (i.e. 2^{B+1}=*even number*), and with even number of levels, we cannot have both:

(i) level of zero amplitude

(ii) equal number of positive and negative levels

simultaneously.

$\Longrightarrow$ But as B gets larger, the difference becomes negligible.

(b) Any coding scheme may be used, but we want to use a binary code that permits us to do *arithmetic* directly with the codeword.

(i.e. each code represents a *scaled*(both in sign and magnitude) expression of the quantized sample.)

$\Longrightarrow$ Typical coding scheme used mostly

: Two's complement binary number system

$$\hat{x}_B[n] = a_0 a_1 a_2 \ldots\ldots a_B$$

where $a_i = 0, 1$ for $i = 0, 1, 2, \cdots, B$, and its *scaled* associated value is:

$$-1 \leq \text{value of } \hat{x}_B[n] = -a_0 \cdot 2^0 + \sum_{k=1}^{B} a_k \cdot 2^{-k} < 1$$

Notice that the value of the summation in above equation is in the range of $[0, 1)$.

(cf.) The MSB a_0 represents the sign of value:

$$a_0 = \begin{cases} 0 & \longrightarrow & + \\ 1 & \longrightarrow & - \end{cases}$$

(c) X_m is called the *full scale level* of A/D converter.
(e.g. $X_m = 10,\ 5,$ or 1 volt)

Corresponding step size Δ is then:

$$\Delta = \frac{2X_m}{2^{B+1}} = \frac{X_m}{2^B}$$

and the relationship between the codeword ($\hat{x}_B[n]$) and the quantized sample $\hat{x}[n]$ is:

$$\hat{x}[n] = X_m \cdot \hat{x}_B[n]$$

Note that since $-1 \leq \hat{x}_B[n] < 1$, we have:

$$-X_m \leq \hat{x}[n] < X_m$$

i.e.:

$\Longrightarrow$ $\hat{x}_B[n]$ is proportional to $\hat{x}[n]$

$\Longrightarrow$ $\hat{x}_B[n]$ is the normalized version of $\hat{x}[n]$

$\Longrightarrow$ $\hat{x}_B[n]$ can be directly used for arithmetic!!!!!

Example 4.6

A/D concersion with 3-bit codeword $\hat{x}_B[n]$:

Figure 4.62: A/D conversion and associated codeword $\hat{x}_B[n]$.

4.7.3 Analysis of the quantization error

During the process of quatization, errors inevitably occurs:

$\Longrightarrow$ called *quantization error* $e[n]$ $\ni$:

$$e[n] \triangleq \hat{x}[n] - x[n]$$

and the quantization error is within the following range:

$$-\frac{\Delta}{2} < e[n] \leq \frac{\Delta}{2} \tag{4.11}$$

Remarks:

(i) (4.11) is valid only when $x[n]$ is within the dynamic range of the A/D converter;

$$-X_m - \frac{\Delta}{2} < x[n] \leq X_m - \frac{\Delta}{2}$$

(ii) Otherwise, $|e[n]| > \frac{\Delta}{2}$ and the sample is called to have been **clipped**.

Mathematical analysis:

The quantization error $e[n]$ is modeled as an additive noise, i.e.

Figure 4.63: Quantization error modeled as an additive noise.

Remarks:

(1) Normally, $e[n]$ is modeled as a *stationary random process* [21], where

$$|e[n]| \leq \frac{\Delta}{2}$$

(2) The fidelity of quantization is usually measured by the SNR at the output $\hat{x}[n](= x[n] + e[n] = \text{signal} + \text{noise})$, i.e.

$$\text{SNR} \triangleq 10 \log_{10} \left(\frac{\sigma_x^2}{\sigma_e^2} \right)$$

where

$$\begin{aligned} \sigma_x^2 : & \quad \text{signal power} \\ \sigma_e^2 : & \quad \text{noise power} \end{aligned}$$

Since we need backgrounds on random (stochastic) processes in order to analyze the quantization error, we *omit* detailed analyses here, but refer tyhe final result as:

$$\sigma_e^2 = \frac{2^{-2B} \cdot X_m}{12}$$

which is the noise power of the $(B+1)$ bit quantizer w/ full scale level of X_m. Therefore, the signal-to-noise ration becomes:

$$\begin{aligned} \text{SNR} &= 10 \log_{10} \left(\frac{12 \cdot 2^{2B} \sigma_x^2}{X_m^2} \right) \\ &= 6.02 \cdot B + 10.8 - 20 \log_{10} \left(\frac{X_m}{\sigma_x} \right) \end{aligned}$$

Note:

(a) The higher SNR is equivalent to the less quantization errors.

(b) As the number of bits (B) increases, SNR increases.

(c) σ_x (rms amplitude of signal) and X_m should be carefully matched to attain high SNR: **IF**

(i) σ_x is too small $(\sigma_x \ll X_m) \longrightarrow$ SNR decreases

(ii) σ_x is too large $(\sigma_x \gg X_m) \longrightarrow$ SNR increases, but clipping occurs i.e. distortion

$\Longrightarrow$ σ_x should be tuned via amplifier before A/D conversion.

[21] Refer Appendix A.

4.7.4 Digital to analog(D/A) conversion

D/A conversion is a physically realizable counterpart to the D/C conversion:

Figure 4.64: The block diagram of D/A conversion.

Since the analog signal $x_{DA}(t)$ from the D/A converter is the output signal of the zero-order hold system whose impulse response is $h_o(t)$ where the input is a impulse train, we have:

$$\begin{aligned}
x_{DA}(t) &= \sum_{n=-\infty}^{\infty} \hat{x}[n]\delta(t-nT) * h_o(t) \\
&= \sum_{n=-\infty}^{\infty} \hat{x}[n]h_o(t-nT) \\
&= \sum_{n=-\infty}^{\infty} (x[n]+e[n])\, h_o(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x[n]h_o(t-nT) + \sum_{n=-\infty}^{\infty} e[n]h_o(t-nT) \\
&\triangleq x_o(t) + e_o(t)
\end{aligned}$$

: signal component + noise component

Figure 4.65: The impulse response $h_o(t)$ and transfer function $H_o(\Omega)$ of zero-order hold system.

Figure 4.66: An example of $x_{DA}(t)$.

Recall: D/C conversion (interpolation or reconstruction)

$$X_r(\Omega) = H_r(\Omega)X\left(e^{j\Omega T}\right) \tag{4.12}$$

Figure 4.67: D/C conversion w/ ideal reconstruction filter $H_r(\Omega)$.

(cf.) Comparing the ideal reconstruction filter $H_r(\Omega)$ with $H_o(\Omega)$ above, we can notice that the above D/A conversion will (or might) cause some serious distortion!!!

Now, consider:

$$\begin{aligned} X_o(\Omega) = \mathcal{F}\left\{x_o(t)\right\} &= \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x[n]h_o(t-nT)\right\} \\ &= \sum_{n=-\infty}^{\infty} x[n]H_o(\Omega)e^{-j\Omega nT} \\ &= H_o(\Omega)\cdot X\left(e^{j\Omega T}\right) \end{aligned} \tag{4.13}$$

Comparing (4.12) and (4.13), we see that $x_{DA}(t)$ should be passed through a *compensated reconstruction filter* $\tilde{H}_r(\Omega)$ defined as:

$$\tilde{H}_r(\Omega) \triangleq \frac{H_r(\Omega)}{H_o(\Omega)}$$

i.e.:

Figure 4.68: Compenstated D/A conversion.

Then, we have: [22]

$$\begin{aligned} \hat{X}_r(\Omega) &= X_{DA}(\Omega)\cdot\tilde{H}_r(\Omega) \\ &= \left[X_o(\Omega)+E_o(\Omega)\right]\cdot\tilde{H}_r(\Omega) \\ &= H_r(\Omega)\cdot X\left(e^{j\Omega T}\right)+E_o(\Omega)\cdot\frac{H_r(\Omega)}{H_o(\Omega)} \\ &\triangleq X_r(\Omega)+E_r(\Omega) \end{aligned}$$

$$\stackrel{\mathcal{F}^{-1}}{\Longrightarrow} \hat{x}_r(t) = x_r(t)+e(t)$$

[22]Refer the equation (4.12), which is: $X_r(\Omega) = H_r(\Omega)X\left(e^{j\Omega T}\right)$.

Note:

(1) If the sampling period T was chosen to satisfy the Nyquist criterion, then $x_r(t) \equiv x_c(t)$.

(2) (Why) *compensated* reconstruction filter: $\tilde{H}_r(\Omega)$?

where

$$H_o(\Omega) = \mathcal{F}\left\{h_o(t)\right\} = \int_0^T e^{-j\omega t} dt = \ldots\ldots = \frac{\sin\left(\frac{\Omega T}{2}\right)}{\frac{\Omega T}{2}} \cdot e^{-j\frac{\Omega T}{2}}$$

Figure 4.69: Ideal reconstruction filter $H_r(\Omega)$ and the zero-order hold system $H_o(\Omega)$.

In order to compensate the non-ideal characteristics of $H_o(\Omega)$, we add another filter $\tilde{H}_r(\Omega)$ such that:

$$\tilde{H}_r(\Omega) = \frac{H_r(\Omega)}{H_o(\Omega)} = \begin{cases} \frac{\Omega T^2}{2} / \sin\left(\frac{\Omega T}{2}\right) \cdot e^{j\frac{\Omega T}{2}}, & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

Figure 4.70: Compenstated reconstruction filter $\tilde{H}_r(\Omega)$.

(cf.) Phase compensation cannot be realized... (see p.126 of the textbook for details.)

Summary;

Considering:

(A) Pre-filtering to avoid aliasing

(B) A/D conversion

(C) Quantization (error)

(D) D/A conversion with compensation

The overall *practical system* for processing continuous signals with discrete system should be in the following form:

Figure 4.71: The practical system for processing continuous signals with discrete system.

4.8 Application of decimation and interpolation to A/D and D/A

In theory, the analog filters ($H_{aa}(\Omega)$ and $H_r(\Omega)$ are required to have very sharp cutoff characteristics. [23]

$\Longrightarrow$ Impractical or very high cost

$\Longrightarrow$ Using decimation and interpolation techniques (in discrete systems), we can loosen the cutoff characteristics requirement (on continuous system), and replace the role of continuous filters with discrete counterparts as well.

$\Longrightarrow$ **Cost effective system design**

Methodology:

(1) High sampling rate (far above Nyquist rate): oversampling

$\longrightarrow$ A very simple lowpass filter $H_{aa}(\Omega)$ can be used, and it can be inexpensive for relatively low bandwidth signals due to possible loose specifications on A/D.

(2) Decimation ($\downarrow M$)

$\longrightarrow$ Computations can be minimized for discrete systems.

(3) Interpolation ($\uparrow L$)

$\longrightarrow$ A very simple reconstruction filter $H_r(\Omega)$ can be used.

Block diagram: [24]

Figure 4.72: A cost effective DSP system.

[23] Continuous(analog) filters contribute the major part of the cost for overall system.

[24] In D/A, the compensated reconstruction filter(continuous) is incorporated into the interpolation filter(discrete).

(cf.) Refer to pp.187 - 188 of the textbook.

Illustration:

Note that T is chosen to be $\frac{\pi}{T} \gg \Omega_N$, i.e. oversampling.
Otherwise, if T was chosen $\ni$: $\frac{\pi}{T} = \Omega_N$, a sharp cutoff for $H_{aa}(\Omega)$ would have been required.

$H_{aa}(\Omega)$

$\Downarrow$

A/D

$\Downarrow$

Decimation($\downarrow M$)

$\Downarrow$

Figure 4.73: An example of cost effective digital signal processing.

DLTI $H\left(e^{j\omega}\right)$

$\Downarrow$

Interpolation($\uparrow L$)

$\Downarrow$

D/A w/ simple reconstruction filter $H_r(\Omega)$

$\Downarrow$

$\Downarrow$

$Y_r(\Omega)$

Figure 4.74: An example of cost effective digital signal processing.(*continued*)