Contents

Chapter 5

Transform Analysis of DLTI Systems

5.1 Introduction

Objective: Analysis of DLTI systems using DTFT and Z-transforms:

Figure 5.1: A DLTI system.

Input/output relationship:

1. Time domain:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

2. Frequency domain and Z-domain:

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) \cdot X\left(e^{j\omega}\right)$$

$$Y(z) = H(z) \cdot X(z)$$

where

$$H\left(e^{j\omega}\right) = F\left\{h[n]\right\} \quad : \textit{frequency response} \text{ of system}$$

$$H(z) = Z\left\{h[n]\right\} \quad : \textit{system function} \text{ of system}$$

5.2 The frequency response of DLTI systems

$$H\left(e^{j\omega}\right) = F\left\{h[n]\right\}$$

The input/output relationship in terms of DTFT for a DLTI system is given by:

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) \cdot X\left(e^{j\omega}\right) \tag{5.1}$$

where

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\Phi_H\left(e^{j\omega}\right)}$$

and we call:

(i) $|H\left(e^{j\omega}\right)|$: magnitude response (or gain)

(ii) $\Phi_H(e^{j\omega})$: phase response

The i/o relationship in (5.1) can be re-written as:

$$\begin{aligned} Y\left(e^{j\omega}\right) &= H\left(e^{j\omega}\right) \cdot X\left(e^{j\omega}\right) \\ &= \left|H\left(e^{j\omega}\right)\right| e^{j\Phi_H\left(e^{j\omega}\right)} \cdot \left|X\left(e^{j\omega}\right)\right| e^{j\Phi_X\left(e^{j\omega}\right)} \\ &= \left|H\left(e^{j\omega}\right)\right| \cdot \left|X\left(e^{j\omega}\right)\right| e^{j\left[\Phi_H\left(e^{j\omega}\right)+\Phi_H\left(e^{j\omega}\right)\right]} \\ &\triangleq \left|Y\left(e^{j\omega}\right)\right| e^{j\Phi_Y\left(e^{j\omega}\right)} \end{aligned}$$

Therefore, we have the magnitude and phase spectra of the output as:

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| \cdot \left|X\left(e^{j\omega}\right)\right|$$

$$\Phi_Y(e^{j\omega}) = \Phi_H(e^{j\omega}) + \Phi_H(e^{j\omega}$$

5.2.1 Ideal frequency selective filters

(A) Ideal LPF:

Figure 5.2: The frequency response of an ideal LPF: $H_{lp}(e^{j\omega})$.

$$H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$: \text{period} = 2\pi$$

The impulse response of an ideal LPF is then:

$$\begin{aligned} h_{lp}[n] = F^{-1}\left\{H_{lp}(e^{j\omega})\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{lp}(e^{j\omega})e^{j\omega n}d\omega \\ &\quad \vdots \quad \text{(assignment)} \\ &= \frac{\sin(\omega_c n)}{\pi n} \\ &\quad : \text{for } -\infty < n < \infty \end{aligned}$$

(B) Ideal HPF:

Figure 5.3: The frequency response of an ideal HPF: $H_{hp}(e^{j\omega})$.

$$H_{hp}(e^{j\omega}) = \begin{cases} 0, & |\omega| < \omega_c \\ 1, & \omega_c < |\omega| \leq \pi \end{cases}$$

$$: \text{period} = 2\pi$$

The impulse response of an ideal HPF is then:

$$\begin{aligned} h_{hp}[n] = F^{-1}\left\{H_{hp}(e^{j\omega})\right\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{hp}(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[1 - H_{lp}(e^{j\omega})\right] e^{j\omega n}d\omega \\ &= \delta[n] - h_{lp}[n] \\ &= \delta[n] - \frac{\sin(\omega_c n)}{\pi n} \\ &\quad : \text{ for } -\infty < n < \infty \end{aligned}$$

Remarks:

(1) Ideal filters above ($h_{lp}[n]$ and $h_{hp}[n]$) are *non-causal*, i.e. $h[n] \neq 0 \ \forall \, n$.
: physically unrealizable

(2) Phase response $\Phi_H(e^{j\omega})$ are assumed to be *zero* $\forall \, \omega$.

$\Longrightarrow$ To make filters to be causal, we have shift the impulse response $h[n]$, and in doing so, non-zero phase response will be introduced... [1]

5.2.2 Phase distortion and delay

Consider an ideal delay system : (n_d samples delay)

Figure 5.4: Ideal delay system.

$$h_{id}[n] = \delta[n - n_d]$$

The frequency response of the ideal delay system is then:

$$\begin{aligned} H_{id}(e^{j\omega}) = F\{h_{id}[n]\} &= \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} \\ &= e^{-j\omega n_d} \end{aligned}$$

where

(i) $|H_{id}(e^{j\omega})| = 1$

(ii) $\Phi_{H_{id}}(e^{j\omega}) = -\omega \cdot n_d, \qquad |\omega| < \pi$

Notice that the system has a **linear phase** characteristics !!!

[1] Recall Signals & Systems class for continuous ideal filters.

Figure 5.5: Magnitude and phase responses of an deal delay system($n_d = 3$).

Remarks:

(1) Linear phase response of a system introduces a *simple delay* on the output sequence

$\Longrightarrow$ considered as a mild (or inconsequential) form of a phase distortion

: *willing to accept*

$\Longrightarrow$ always can be compensated by introducing another delay in other parts of the overall system.

(2) Introducing (ideal) delay on ideal filters, system can be made *approximately causal*, i.e.

(e.g.)

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Corresponding impulse response is then:

$$h_{lp}[n] = \frac{\sin[\omega_c(n - n_d)]}{\pi(n - n_d)}$$

(cf.) We can never make ideal filters exactly causal!

5.2.3 Group delay

:Measure of the linearity of phase

Definition 5.1 The group delay $\tau(\omega)$ of a DLTI system with frequency response $H(e^{j\omega})$ is defined by:

$$\begin{aligned}\tau(\omega) &= \text{grd}\left[H(e^{j\omega})\right] \\ &\triangleq -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\} \\ &: \quad \text{negative slope of phase response at } \omega\end{aligned}$$

where

$$\arg\left[H(e^{j\omega})\right] = \Phi_H(e^{j\omega}), \quad \text{for } 0 < \omega < \pi$$

Note:

(1) If $\tau(\omega) =$ constant, the system has a linear phase, i.e. an ideal delay, and there does not exists any significant distortion at the output.

(2) Deviation of $\tau(\omega)$ away from a constant represents the degree of non-linearity of the phase.

Example 5.1

Ideal delay system:

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$\Longrightarrow \arg\left[H_{id}(e^{j\omega})\right] = -\omega n_d, \qquad 0 < \omega < \pi$$

$$\Longrightarrow \tau(\omega) = -\frac{d}{d\omega}\{-\omega n_d\} = n_d : \text{constant}$$

5.3 System function described by linear constant coefficient difference equation

Consider a DLTI system with input/output relation described by:

Figure 5.6: DLTI system.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{5.2}$$

Taking the Z-transform of (5.2), we get: [2]

$$\sum_{k=0}^{N} a_k Y(z) z^{-k} = \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$\Longrightarrow \left(\sum_{k=0}^{N} a_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z)$$

The system function $H(z)$ is then given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{5.3}$$

$$\underset{=}{\text{or}} \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \quad : \text{factored form} \tag{5.4}$$

Remarks:

(1) poles : $z = 0$ and $\{z = d_k\}_{k=1}^{N}$
zeros : $z = 0$ and $\{z = c_k\}_{k=1}^{M}$

(2) coeff. in numerator of (5.3) comes from RHS of (5.2): coeff. of $x[n-k]$.
coeff. in denominator of (5.3) comes from LHS of (5.2): coeff. of $y[n-k]$.

[2] Notice that we have applied the linearity and the time shift properties of the Z-transform.

Example 5.2

Given the system function of a DLTI system as:

$$\begin{aligned} H(z) &= \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} \\ &= \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

$$\Longrightarrow \left(1+\tfrac{1}{4}z^{-1}-\tfrac{3}{8}z^{-2}\right)Y(z) = \left(1+2z^{-1}+z^{-2}\right)X(z)$$

$$\overset{Z^{-1}}{\Longrightarrow} y[n]+\tfrac{1}{4}y[n-1]-\tfrac{3}{8}y[n-2] = x[n]+2x[n-1]+x[n-2]$$

: Linear comstant coeff. difference equation of the system.

5.3.1 Stability and causality

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$\Longrightarrow$ $X(z)$ and $Y(z)$ should have overlapping region of ROC for $H(z)$ to be *valid*.

$\overset{\text{but}}{\Longrightarrow}$ ROC of $H(z)$ is not specified yet. (R_H)

$\Longrightarrow$ Depending on R_H, different forms of impulse response $h[n]$ are possible. (even if the system function $H(z)$ is same.)

Restriction of $H(z)$:

(1) If the system is to be **causal**, then $h[n]$ must be a right sided sequence, and thus R_H *must be outside of the outermost pole.*

(2) If the system is to be **stable**, then $h[n]$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$\Longrightarrow$ $\sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$ for $|z| = 1$

$\Longrightarrow$ since $\sum_{n=-\infty}^{\infty} h[n]z^{-n} < \sum_{n=-\infty}^{\infty} |h[n]z^{-n}| < \infty$,

$H(z)$ for $|z| = 1$ must converge

$\Longrightarrow$ *unit circle must be within* R_H

Example 5.3

Consider a DLTI system with the input/output relationship of:

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

Then the system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$

$$= \frac{-\frac{1}{3}}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1 - 2z^{-1}}$$

Figure 5.7: Pole-zero diagram of the given DLTI system.

There ∃ three possible cases of ROC: R_H

(1) $R_H : \; |z| > 2$:

Figure 5.8: $R_H : \; |z| > 2$.

The impulse response $h[n]$ is a right-sided sequence, and thus:

$$h[n] = \left\{ -\frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{4}{3}(2)^n \right\} u[n]$$

(i) causal
(ii) unstable (since the unit circle is not in R_H.) [3]

(2) $R_H : \; |z| < \frac{1}{2}$:

Figure 5.9: $R_H : \; |z| < \frac{1}{2}$.

The impulse response $h[n]$ is a left-sided sequence, and thus:

$$h[n] = \left\{ \frac{1}{3}\left(\frac{1}{2}\right)^n - \frac{4}{3}(2)^n \right\} u[-n-1]$$

(i) non-causal
(ii) unstable (since the unit circle is not in R_H.) [4]

[3] Note that term $\frac{4}{3}(2)^n$ is unstable, since $n \geq 0$.
[4] Note that term $\frac{1}{3}\left(\frac{1}{2}\right)^n$ is unstable, since $n \leq -1$.

(3) $R_H : \frac{1}{2} < |z| < 2$:

Figure 5.10: $R_H : \frac{1}{2} < |z| < 2$.

The impulse response $h[n]$ is a two-sided sequence, and thus:

$$h[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

(i) non-causal (since it is an infinite sequence)
(ii) stable (since the unit circle is inside of R_H.)

Remark:

For a DLTI system to be both bf causal and **stable**;

$\Longrightarrow$ ROC of $H(z)$ must be outside of the outermost pole. (*causality*)
& unit circle must be inside the ROC of $H(z)$. (*stability*)

$\Longrightarrow$ **All of the poles in H(z) must be inside of the unit circle !!!**

5.3.2 Inverse systems

Recall that the inverse system of a DLTI system ($h[n]$) is defined as another DLTI system with impulse response $h_i[n] \ni$:

$$h[n] * h_i[n] = \delta[n]$$

$$\stackrel{Z}{\Longrightarrow} H(z)H_i(z) = 1$$

$$\Longrightarrow H_i(z) = \frac{1}{H(z)}$$

If the DTFT $H(e^{j\omega})$ exists, then the frequency response of the inverse system $H_i(e^{j\omega})$ is given by:

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

Note:

(1) $\log_{10}|H_i(e^{j\omega})| = -\log_{10}|H(e^{j\omega})|$: log magnitude

(2) $\Phi_{H_i}(e^{j\omega}) = -\Phi_H(e^{j\omega})$: phase response

(3) $\tau_i(\omega) = -\tau(\omega)$: group delay

(cf.) Not all DLTI systems have their inverse systems, i.e. if $H(e^{j\omega}) = 0$ for some ω, e.g. ideal LPF, then there does NOT $\exists$ $H_i(e^{j\omega})$.

Let

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

then

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})}{\prod_{k=1}^{M}(1 - c_k z^{-1})}$$

Remarks:

(1) Poles (seros) of $H(z)$ become zeros (poles) of $H_i(z)$.

(2) Since $H(z)H_i(z) = 1$, the ROC of $H(z)$ and ROC of $H_i(z)$ must have *overlap region.*

Example 5.4

Let [5]

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, \qquad \text{ROC}(R_H) \ : \ |z| > 0.9$$

Find the impulse response of the inverse system.

Solution:

The transfer function of the inverse system is:

$$H_i(z) = \frac{1}{H(z)} = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} \quad \text{ROC}(R_{H_i})?$$

Figure 5.11: Pole-zero diagram of $H(z)$ and $H_i(z)$.

[5] Corresponding impulse response is $h[n] = (0.9)^n u[n] - (0.5)(0.9)^{n-1} u[n-1]$, since it is r.s.s..

Among two possible cases ($|z| > 0.5$ or $|z| < 0.5$) for the ROC R_{H_i} of $H_i(z)$, only $|z| > 0.5$ *overlaps* with the ROC R_H of $H(z)$.

$\Longrightarrow$ ROC R_{H_i} : $|z| > 0.5$.

$\Longrightarrow$ $h_i[n] = (0.5)^n u[n] - 0.9(-.5)^{n-1} u[n-1]$

$\Longrightarrow$ The inverse system is both *causal* and *stable*, since the unit circle $\ni$ R_{H_i}.

Example 5.5

$$H(z) = \frac{-0.5 + z^{-1}}{1 - 0.9z^{-1}}, \qquad |z| > 0.9$$

Solution:

The transfer function of the inverse system is:

$$H_i(z) = \frac{1 - 0.9z^{-1}}{-0.5 + z^{-1}} = \frac{-2 + 1.8z^{-1}}{1 - 2z^{-1}}$$

Figure 5.12: Pole-zero diagram of $H_i(z)$ with two possible ROC's.

(i) ROC R_{H_i} : $|z| > 2$

$$h_i[n] = -2(2)^n u[n] + 1.8(2)^{n-1} u[n-1] \quad : \text{causal and unstable}$$

(ii) ROC R_{H_i} : $|z| < 2$

$$h_i[n] = 2(2)^n u[-n-1] - 1.8(2)^{n-1} u[-n] \quad : \text{non-causal and stable}$$

Remarks:

(1) Let $H(z)$ be a causal system with zeros $\{c_k\}_{k=1}^M$, then $H_i(z)$ is also *causal* **iff** the ROC of $H_i(z)$ is given by: [6]

$$|z| > \max_k |c_k|$$

(2) For $H_i(z)$ to be *stable* system as well, the unit circle must be within the ROC, and thus:

$$\max_k |c_k| < 1$$

i.e. all of the poles $\{c_k\}_{k=1}^M$ of $H_i(z)$ are inside of the unit circle.

FACT:

A DLTI system $H(z)$ and its inverse system $H_i(z)$ are both **causal and stable** if and only if all of the *zeros* [7] and *poles* [8] of $H(z)$ are inside the unit circle.

[6] Note that $\{c_k\}_{k=1}^M$ correspond to the poles of $H_i(z)$.
[7] For $H_i(z)$.
[8] For $H(z)$.

5.3.3 Impulse response for rational system function

Recall that the system function for a DLTI system described by a linear, constant coefficient difference equation is given by:

$$
\begin{aligned}
H(z) &= \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})} \\
&= \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad (5.5) \\
&\quad (\text{if } M \geq N)
\end{aligned}
$$

Assuming the system is causal, the impulse response is then:

$$h[n] = \sum_{r=0}^{M-N} B_r \delta[n - r] + \sum_{k=1}^{N} A_k (d_k)^n u[n]$$

Remark:

(1) In (5.5), if at least one non-zero pole (d_k) is NOT canceled by a zero (d_k), [9] then the impulse response $h[n]$ will be of *infinite length.*

$\Longrightarrow$ called an infinite inpulse response (IIR) system.

(2) In (5.5), if $N = 0$ (i.e. all of non-zero poles at d_k are canceled, and there $\exists$ NO pole except at $z = 0$), then $h[n]$ is of *finite length.*

$\Longrightarrow$ called a finite inpulse response (FIR) system, i.e.

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n - k] = \begin{cases} b_k, & 0 \leq n \leq M \\ 0, & \text{elsewhere} \end{cases}$$

$$y[n] = h[n] * x[n] = \sum_{k=0}^{M} b_k \delta[n - k] * x[n] = \sum_{k=0}^{M} b_k x[n - k] \equiv \sum_{k=0}^{M} h[k] x[n - k]$$

[9]That is, at least one term in the form of $A_k(d_k)^n u[n]$ remains.

Example 5.6

Given a causal system with I/O relation of:

$$y[n] - ay[n-1] = x[n]$$

Taking the Z-transform, we get:

$$Y(z)(1 - az^{-1}) = X(z)$$

Therefore, the system function becomes;

$$H(z) = \frac{1}{1 - az^{-1}}$$

Taking the inverse Z-transform, we get the inpulse response $h[n]$ as:

$$h[n] = a^n u[n]$$

Figure 5.13: The impulse response of an IIR system.

Figure 5.14: Pole-zero diagram of $H(z)$ with ROC for $h[n]$ to be causal.

(cf.)

(i) For the system to be stable as well, it should be $|a| < 1$.

(ii) Notice that $h[n]$ is of infinite length.

Example 5.7

Consider a FIR system as follows:

$$h[n] = \begin{cases} a^n, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$$

Taking the Z-transform, we get the system function as:

$$\begin{aligned} H(z) = \sum_{n=0}^{M} a^n z^{-n} = \sum_{n=0}^{M} \left(az^{-1}\right)^n &= \frac{1-(az^{-1})^{M+1}}{1-az^{-1}} \\ &= \frac{z - \frac{a^{M+1}}{z^M}}{z-a} \\ &= \frac{1}{z^M}\frac{z^{M+1}-a^{M+1}}{z-a} \\ &\equiv \frac{Y(z)}{X(Z)} \end{aligned}$$

Figure 5.15: Pole-zero diagram of $H(z)$ for the case of $M = 7$.

Expressing in terms of $X(z)$ and $Y(z)$, we have:

$$(1-az^{-1})Y(z) = \left(1 - a^{M+1}z^{-(M+1)}\right) X(z)$$

Taking the inverse Z-transform, we get:

$$y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1] \tag{5.6}$$

Or, using the given impulse response $h[n]$ and computing the convolution sum, we get another expression of the output sequence as:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = \sum_{k=0}^{M} a^k x[n-k] \tag{5.7}$$

Recall:

The representation of a DLTI system with constant coefficient linear difference equation is *NOT unique.* (refer (5.6) and (5.7).)

5.4 Frequency response for rational system functions

Consider a DLTI system with input/output relationship described by a linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Then, the frequency response of the system is:

$$\begin{aligned} H\left(e^{j\omega}\right) &= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \\ &\overset{\text{or}}{\equiv} \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k e^{-j\omega})}{\prod_{k=1}^{N}(1-d_k e^{-j\omega})} \end{aligned} \tag{5.8}$$

5.4.1 System characteristics

(1) Magnitude response: (log magnitude in dB) [10]

From (5.8), we have:

$$\begin{aligned} 20\log_{10}\left|H(e^{j\omega})\right| &= \text{gain in dB} \\ &= 20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}\left|1-c_k e^{-j\omega}\right| - \sum_{k=1}^{N} 20\log_{10}\left|1-d_k e^{-j\omega}\right| \end{aligned}$$

[10] dB = decibel.

Remarks:

(a) If $|H(e^{j\omega})| < 1$, then $20\log_{10}|H(e^{j\omega})| < 0$, and thus $-20\log_{10}|H(e^{j\omega})| < 0$ corresponds to the *attenuation*, i.e.:

$$\begin{aligned}\text{attenuation in dB} &= -20\log_{10}\left|H(e^{j\omega})\right| \\ &= \text{- gain in dB}\end{aligned}$$

(b) Another advantage of log magnitude:

The magnitude ot the ourput in a DLTI system can be expressed in asimple summation form rather than in a multiplicative form, i.e.: [11]

$$\left|Y(e^{j\omega})\right| = \left|H(e^{j\omega})\right| \cdot \left|X(e^{j\omega})\right|$$

$$\longrightarrow\ 20\log_{10}\left|Y(e^{j\omega})\right| = 20\log_{10}\left|H(e^{j\omega})\right| + 20\log_{10}\left|X(e^{j\omega})\right|$$

(2) Phase response:

From (5.8), we also have: [12]

$$\Phi_H(e^{j\omega}) = \text{phase}\left(\frac{b_0}{a_0}\right) + \sum_{k=1}^{M}\text{phase}\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N}\text{phase}\left(1 - d_k e^{-j\omega}\right)$$

(3) Group delay:

The group delay of the system function is:

$$\begin{aligned}\text{grd}\left[H(e^{j\omega})\right] &= -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\} \\ &= -\sum_{k=1}^{M}\frac{d}{d\omega}\left\{\arg\left(1 - c_k e^{-j\omega}\right)\right\} + \sum_{k=1}^{N}\frac{d}{d\omega}\left\{\arg\left(1 - d_k e^{-j\omega}\right)\right\} \\ &= \sum_{k=1}^{M}\frac{|c_k|^2 - \text{Re}\left\{c_k e^{-j\omega}\right\}}{1 + |c_k|^2 - 2\text{Re}\left\{c_k e^{-j\omega}\right\}} - \sum_{k=1}^{N}\frac{|d_k|^2 - \text{Re}\left\{d_k e^{-j\omega}\right\}}{1 + |d_k|^2 - 2\text{Re}\left\{d_k e^{-j\omega}\right\}}\end{aligned}$$

[11] (cf.) Inverse system.

[12] Here, the term phase $\left(1 - c_k e^{-j\omega}\right)$ is due to *zero* and has *positive effect*, whereas the term phase $\left(1 - d_k e^{-j\omega}\right)$ is due to *pole* and has *negative effect*.

Check: assignment [13]

Hint: [14] $\frac{d}{dx}\left[\arctan\{f(x)\}\right] = \frac{1}{1+f^2(x)}\frac{df}{dx}$.

First, let us define abbreviated notation as follows:

$$\begin{aligned} \arg\left(1-\alpha_k e^{-j\omega}\right) &= \arg\left[1-\mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\} - j\mathrm{Im}\left\{\alpha_k e^{-j\omega}\right\}\right] \\ &\overset{\text{let}}{=} \arg\left[1-\mathrm{Re}-j\mathrm{Im}\right] \\ &= \tan^{-1}\left[\frac{-\mathrm{Im}}{1-\mathrm{Re}}\right] \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \alpha_k e^{-j\omega} &= \left(\alpha_R + j\alpha_I\right)\left(\cos(\omega) - j\sin(\omega)\right) \\ &= \left[\alpha_R\cos(\omega) + \alpha_I\sin(\omega)\right] + j\left[\alpha_I\cos(\omega) - \alpha_R\sin(\omega)\right] \\ &= \mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\} + j\mathrm{Im}\left\{\alpha_k e^{-j\omega}\right\} \end{aligned}$$

From which we get:

$$\frac{d}{d\omega}\left[\mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\}\right] = -\alpha_R\sin(\omega) + \alpha_I\cos(\omega) \equiv \mathrm{Im}\left\{\alpha_k e^{-j\omega}\right\} \tag{5.10}$$

$$\frac{d}{d\omega}\left[\mathrm{Im}\left\{\alpha_k e^{-j\omega}\right\}\right] = -\alpha_R\sin(\omega) - \alpha_I\cos(\omega) \equiv -\mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\} \tag{5.11}$$

[13] The textbook has some typo error on this equation.

[14] Let $y = \tan^{-1}(x) \longrightarrow \tan(y) = x \longrightarrow \sec^2(y)dy = dx \longrightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$.

Plugging (5.10) and (5.11) into (5.9), we have:

$$\begin{aligned}
\operatorname{grd}\left(1-\alpha_k e^{-j\omega}\right) &= -\frac{d}{d\omega}\left\{\arg\left(1-\alpha_k e^{-j\omega}\right)\right\} \\
&= -\frac{d}{d\omega}\left\{\tan^{-1}\left[\frac{-\mathrm{Im}}{1-\mathrm{Re}}\right]\right\} \\
&= -\left\{\frac{1}{1+\frac{\mathrm{Im}^2}{(1-\mathrm{Re})^2}}\cdot\frac{\mathrm{Re}(1-\mathrm{Re})+\mathrm{Im}(-\mathrm{Im})}{(1-\mathrm{Re})^2}\right\} \\
&= -\left\{\frac{(1-\mathrm{Re})^2}{(1-\mathrm{Re})^2+\mathrm{Im}^2}\cdot\frac{\mathrm{Re}-(\mathrm{Re}^2+\mathrm{Im}^2)}{(1-\mathrm{Re})^2}\right\} \\
&= -\frac{\mathrm{Re}-(\mathrm{Re}^2+\mathrm{Im}^2)}{1-2\mathrm{Re}+(\mathrm{Re}^2+\mathrm{Im}^2)} \\
&= -\frac{\mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\}-|\alpha_k|^2}{1-2\mathrm{Re}\left\{\alpha_k e^{-j\omega}\right\}+|\alpha_k|^2}
\end{aligned}$$

Note:
The (1) log magnitude, (2) phase response, and the (3) group delay are all in the form of **summation** by the contributions from each pole and zero of the system function !

Remarks:

(a) Recall that the principal value (between $-\pi$ and π) of the phase response $\Phi_H(e^{j\omega})$ is denoted by $\text{ARG}\left[H(e^{j\omega})\right]$, i.e.:

$$-\pi < \text{ARG}\left[H(e^{j\omega})\right] \leq \pi$$

$\Longrightarrow$ $\Phi_H(e^{j\omega}) = \text{ARG}\left[H(e^{j\omega})\right] + 2\pi r(\omega)$ where $r(\omega)$ is an integer dependeing on the frequency ω, and it could be *discontinuous*.

$\Longrightarrow$ Typically, the phase response of a system is represented by $\text{ARG}\left[H(e^{j\omega})\right]$.

Example 5.8

Recall that:

$$\arg\left[H(e^{j\omega})\right] \triangleq \Phi_H(e^{j\omega}) \;\; \text{for } 0 < \omega \leq \pi$$

Figure 5.16: Comparison between $\arg\left[H(e^{j\omega})\right]$ and $\text{ARG}\left[H(e^{j\omega})\right]$ w/ corresponding $r(\omega)$.

(b) From the phase response characteristics, we have: [15]

$$\begin{aligned}
&\text{ARG}\left[H(e^{j\omega})\right] \\
= \;& \text{ARG}\left(\frac{b_0}{a_0}\right)\sum_{k=1}^{M}\text{ARG}\left(1 - c_k e^{-j\omega}\right) - \sum_{k=1}^{N}\text{ARG}\left(1 - d_k e^{-j\omega}\right) + 2\pi r(\omega)
\end{aligned}$$

OR

$$\text{ARG}\left[H(e^{j\omega})\right] = \tan^{-1}\left\{\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right\} \quad : \text{principal value}$$

$$\text{where} \quad H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

(c) The group delay of the system:

$$\begin{aligned}
\text{grd}\left[H(e^{j\omega})\right] &= -\frac{d}{d\omega}\left\{\arg\left[H(e^{j\omega})\right]\right\} \\
&= -\frac{d}{d\omega}\left\{\text{ARG}\left[H(e^{j\omega})\right]\right\}
\end{aligned}$$

: except for dixcontinuities (i.e., impulses)

(d) Note that the log magnitude, phase, and group delay of the system are represented **as a sum of contributions from each pole and zero of the system function**.

[15] Here, the term $2\pi r(\omega)$ is included in order to make $-\pi < \text{ARG}\left[H(e^{j\omega})\right] \leq \pi$.

5.4.2 Frequency response of a single pole or zero

[16]

Recall that the contribution of each pole and zero on the magnitude, phase, and group delay of the system function (or frequency response) is in the following form:

$$\begin{aligned} 1 - \alpha_k e^{-j\omega} &= 1 - re^{j\theta}e^{-j\omega} \\ &= 1 - re^{j(\theta-\omega)} \\ &= 1 - r\cos(\theta-\omega) - j\sin(\theta-\omega) \end{aligned}$$

where pole (zero) : $\alpha_k = re^{j\theta}$.

Figure 5.17: Representation of α_k in a polar coordinate.

[16]Be reminded that the pole has a *substractive* effect, whereas the zero has a *additive* effect.

(1) Log magnitude:

$$\begin{aligned}
20\log_{10}\left|1-\alpha_k e^{-j\omega}\right| &= 20\log_{10}\left|1-re^{j(\theta-\omega)}\right| \\
&= 20\log_{10}\left[\{1-r\cos(\theta-\omega)\}^2+r^2\sin(\theta-\omega)\right]^{\frac{1}{2}} \\
&= 10\log_{10}\left[1-2r\cos(\theta-\omega)+r^2\right] \\
&= 10\log_{10}\left[1-2r\cos(\omega-\theta)+r^2\right] \qquad (5.12)
\end{aligned}$$

(2) Phase:

$$\begin{aligned}
\text{ARG}\left[1-\alpha_k e^{-j\omega}\right] &= \tan^{-1}\left[\frac{-r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right] \\
&= \tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right] \qquad (5.13)
\end{aligned}$$

(3) Group delay:

$$\begin{aligned}
\text{grd}\left[1-\alpha_k e^{-j\omega}\right] &= \frac{\text{Re}\left\{|\alpha_k|^2-\alpha_k e^{-j\omega}\right\}}{1-2\text{Re}\left\{\alpha_k e^{-j\omega}\right\}+|\alpha_k|^2} \\
&= \frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)} \qquad (5.14) \\
&= \frac{r^2-r\cos(\omega-\theta)}{|1-r^{j\theta}e^{-j\omega}|^2}
\end{aligned}$$

Example 5.9

Log magnitude $20\log_{10}\left|1-re^{j\theta}e^{-j\omega}\right|$, phase $\text{ARG}\left[1-re^{j\theta}e^{-j\omega}\right]$, and group delay $\text{grd}\left[1-re^{j\theta}e^{-j\omega}\right]$:

solid line : $\theta = 0$, dotted line : $\theta = \pi$, where $r = 0.9$

Figure 5.18: Log magnitude, phase, and group delay.

(cf.) Notice that dotted line is a shifted version of solid line in amount of $\omega = \theta$.

Remarks:

(a) Note that (5.12), (5.13), and (5.14) are periodic in ω with period of 2π.

(b) $(5.12) = 20\log_{10}\left|1-re^{j\theta}e^{-j\omega}\right|$:

(i) It dips at $\omega = \theta$ [17]. (will be explined later...)

(ii) The maximum of (5.12) occurs at $\theta - \omega = \pi$, where for $r = 0.9$:

$$\max\{(5.12)\} = 10\log_{10}\left(1+2r+r^2\right) = 20\log_{10}(1+r) \approx 5.57(\text{dB})$$

(iii) The minimum of (5.12) occurs at $\theta - \omega = 0$, where for $r = 0.9$:

$$\min\{(5.12)\} = 10\log_{10}\left(1-2r-r^2\right) = 20\log_{10}|1-r| \approx -20(\text{dB})$$

(c) $(5.13) = \text{ARG}\left[1-re^{j\theta}e^{-j\omega}\right] = 0$ at $\omega = \theta$.

(d) $(5.14) = \text{grd}\left[1-re^{j\theta}e^{-j\omega}\right]$

high positive slope at $\omega = \theta$ in (5.13) $\equiv$ large negative peak at $\omega = \theta$ in (5.14)

[17] Maximum attenuation or minimum gain.

5.4.3 Geometric interpretations on the factor $1 - re^{j\theta}e^{-j\omega}$

: Useful for *approximate sketching* of $H(e^{j\omega})$ directly from pole-zero diagram of $H(z)$.

Consider a first order system function $H(z)$ as follows: [18]

$$H(z) = \left(1 - re^{j\theta}z^{-1}\right) = \frac{z - re^{j\theta}}{z}, \quad r < 1$$

Then, the frerquency response is:

$$H(e^{j\omega}) = \left(1 - re^{j\theta}e^{-j\omega}\right) = \frac{e^{j\omega} - re^{j\theta}}{e^{j\omega}}$$

$$\text{pole} : z = 0, \ \text{zero} : z = re^{j\theta}$$

Figure 5.19: Pole-zero diagram.

where the pole vector [19] $\vec{v_1}$, $\vec{v_2}$, and the zero vector [20] $\vec{v_3}$ are given recpectively as:

(1) $\vec{v_1} = e^{j\omega}$
(2) $\vec{v_2} = re^{j\theta}$
(3) $\vec{v_3} = \vec{v_1} - \vec{v_2} = e^{j\omega} - re^{j\theta}$ phase $= \phi_3$

(1) Magnitude: [21]

$$\left|H(e^{j\omega})\right| = \left|1 - re^{j\theta}e^{-j\omega}\right| = \frac{|e^{j\omega} - re^{j\theta}|}{|e^{j\omega}|} = \frac{|\vec{v_3}|}{|\vec{v_1}|} = |\vec{v_3}|$$

(2) Phase: [22]

$$\Phi_H(e^{j\omega}) = \text{phase}\,(\vec{v_3}) - \text{phase}\,(\vec{v_1}) = \phi_3 - \omega$$

[18] Condition $r < 1$ os for stability of the system.
[19] Vector from a pole to a point on the unit circle.
[20] Vector from a zero to a point on the unit circle.
[21] Magnitude of zero vector.
[22] Phase of zero vector minus phase of pole vector.

REMARKS: Contribution of $\left(1 - re^{j\theta}e^{-j\omega}\right)$

(a) Magnitude:

$$|\vec{v_3}| = |\text{vector from a zero to a point } z = e^{j\omega}|$$

$\longrightarrow$ $\text{argmin}_\omega |\vec{v_3}| = \theta$ from the plot at previous page.

$\longrightarrow$ so $|\vec{v_3}|$ is minimum when $\omega = \theta$.

$\longrightarrow$ explains the sharp dip of log magnitude at $\omega = \theta$.

(b) Phase:

$$\begin{aligned} \Phi_H(e^{j\omega}) &= \text{phase}\,(\vec{v_3}) - \text{phase}\,(\vec{v_1}) = \phi_3 - \omega \\ &= \text{phase(zero vector)} - \text{phase(pole vector)} \end{aligned}$$

Example 5.10

$$\theta = \pi \ \text{ (refer the plot at example5.9.)}$$

$$\omega_2 < \omega_1, \ \text{and } \phi_3 - \omega \le 0 \ \text{ (where } 0 \le \omega \le \pi)$$

Figure 5.20: Pole vector and zero vector.

As ω increases ($\omega = 0 \rightarrow \omega_1 \rightarrow \omega_2 \rightarrow \pi$), $\Phi_H(e^{j\omega})$ starts from **zero**, and the difference gets larger, whereas as $\omega \rightarrow \pi$, the difference becomes smaller and approaches to **zero** again. (see plot at example5.9.)

(c) Effect of radius r:

A. As $r \rightarrow 1$, we have: [23]

(i) Log magnitude dips more sharply at $\omega = \theta$, and becomes $-\infty$ (dB) at $\omega = \theta$ when $r = 1$.

(ii) The positive slope of phase function *around* $\omega = \theta$ becomes ∞ ar $r \rightarrow 1$, and becomes discontinuous at $\omega = \theta$ when $r = 1$.

(iii) Since group delay is negative slope of the phase function, group delay is negative around $\omega = \theta$, and dips more sharply ar $r \rightarrow 1$. And away from $\omega = \theta$, group delay has a relatively flat positive values.

$\Longrightarrow$ **self study:** p.221 of your textbook

B. For $r > 1$, we have:

(iv)
i. Magnitude response has similar characteristics of the case when $r \leq 1$.
ii. Phase has negative slope for all ω, and has discontinuities at $\omega = \theta$.
iii. Group delay has positive values for all ω.

$\Longrightarrow$ **self study:** p.223 of your textbook

(d) If $\left(1 - re^{j\theta}e^{-j\omega}\right)$ is a pole factor, i.e. $1/\left(1 - re^{j\theta}e^{-j\omega}\right)$, all of the characteristics become *negative* of previously discussed characteristics. ((cf.) Inverse system.)

Examples pp.225 $\sim$ 230 : Self study

[23] Refer the figure5.8 at p.222 of your textbook.