Contents

Chapter 9

INTRODUCTION TO RANDOM PROCESSES

9.1 Introduction

Recall: A r.v. is a function mapping each point in S to a point in R^1.

$\Longrightarrow$ A *random process* is a function mapping each point in S to a function of time.

(e.g.)

1. A random variable: $X(\zeta)$
2. A random process: $X(t, \zeta)$

Figure: Illustration of the concept of a r.p..

Remark:

1. For a given $\zeta = \zeta_1$, we get a specific time function: called a *sample function* $X(t, \zeta_1) \stackrel{\text{d}}{=} x_1(t)$
2. For a fixed time $t = t_1$ from an index set I, we get a random variable: $X(t_1, \zeta) \stackrel{\text{d}}{=} X_1$
3. If both t and ζ are fixed, we get a constant: $X(t_1, \zeta_1) = \textit{constant}$
4. The totality of the sample function is called the *ensemble*
5. We usually denote the r.p. as $X(t)$ WLOG

Definition 9.1 Random Process*:*
A r.p. is a family of random variables: $\{X_1, X_2, X_3, \cdots\}$

Category of r.p.: $X(t, \zeta)$

(1) *Discrete random sequence*: t and X are both discrete

(2) *Continuous random sequence*: t is discrete and X is continuous

(3) *Continuous random process*: t and X are continuous

(4) *Discrete random process*: t is continuous and X is discrete

(cf) Mostly, we deal w/ type (3) and (4), i.e. continuous random process and discrete random process.....

9.2 Examples of Random Processes

Example 9.1 *Discrete random sequence:* discrete-time/discrete-valued

Let $I = \{0, 1, 2, \ldots\}$, and at each $t_k \in I$ a fair coin is flipped. If a head comes up, amplitude value of 1 is assigned, whereas amplitude -1 is assigned if a tail is obtained. (figure 9-1)

Figure 9.1: A discrete random sequence.(sample functions)

Example 9.2 *Continuous random sequence:* discrete-time/continuous-valued

A r.p described by the following recurrence relationship:

$$X_{k+1} = X_k + e_k, \quad k = 0, 1, 2, \cdots$$

where $\{e_0, e_1, \cdots\}$ is a sequence of mutually independent r.v.'s.
Sample functions are shown in figure 9-2 for the case of $e_k \sim U[-1, 1]$.

Figure 9.2: A continuous random sequence.(sample functions)

Example 9.3 *Continuous random process:* continuous-time/continuous-valued

A family of waveforms expressed as:

$$X(t, \Theta) = A\cos(\omega_0 t + \Theta)$$

where A and ω_0 are constants, Θ is a r.v..

Sample functions are shown in figure 9-3 for the case of $\Theta \sim U[0, 2\pi]$.

(cf) Although the sample functions look deterministic, it is a r.p. since the phase is random: observe the *ensemble* for a fixed t.

Figure 9.3: A continuous random process.(sample functions)

Example 9.4 *Discrete random process:* continuous-time/discrete-valued

An example of the discrete r.p. can be obtained by *hard limiting* the $X(t, \Theta)$ in example 9-3 by setting the amplitude to 1 if greater than zero, and to -1 if not. (Sample functions are shown in figure 9-4 for this case.)

Figure 9.4: A discrete random process.(sample functions)

9.3 Statistical Descriptions of Random Processes

9.3.1 Probability Density Functions

Definition 9.2 The first order p.d.f. at a given time t_1:
The first order pdf $f_X(x, t_1)$ of a r.p. $X(t)$ at time $t = t_1$ is defined as the pdf of the r.v. $X(t_1) \triangleq X_1$ of which interpretation is as follows:

$$f_X(x, t)dx = P[x < X(t) \leq x + dx \textit{ at time } t]$$

Definition 9.3 The joint p.d.f. at two time instances t_1 and t_2:
The joint pdf $f_{X_1X_2}(x_1, t_1; x_2, t_2)$ of a r.p. $X(t)$ at times $t = t_1$ and $t = t_2$ is defined as the joint pdf of two r.v.'s $X(t_1) \triangleq X_1$ and $X(t_2) \triangleq X_2$ of which interpretation is as follows:

$$f_{X_1,X_2}(x_1, t_1; x_1, t_2)dx_1dx_2 = P[x_1 < X(t_1) \leq x_1 + dx_1 \textit{ and } x_2 < X(t_2) \leq x_2 + dx_2]$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

$\Longrightarrow$ A complete statistical description of a r.p. requires the joint pdf at N arbitrary time instants...

$\Longrightarrow$ Needs tremendous amount od data (except fo special case $\ni$: Gaussian r.p.)

$\Longrightarrow$ Often we settle for less, in the way of statistical description (e.g. certain averages)

9.3.2 Ensemble Averages

The averages for a r.p. are obtained by the pdf's discussed just before.....

Definition 9.4 The first order average of $X(t)$:
The first order averages are obtained using the first order pdf. Let $g[X(t)]$ be some function of the r.p. $X(t)$, then the average of it is defined as:

$$E\{g[X(t)]\} = \int_{-\infty}^{\infty} g(x) f_X(x,t) dx$$

(cf) Note that $E\{g[X(t)]\}$, in general, is a function of time t...

<u>**Examples**</u> [1]

(1) Mean of a r.p.: $E\{X(t)\}$

(2) Mean square of a r.p.: $E\{X^2(t)\}$

Extending the idea to two time instances, we have:

Definition 9.5 The second order average of $X(t)$:
The 2nd order averages are obtained using the second order pdf.

$$E\{h[X(t_1), X(t_2)]\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x_1, x_2) f_{X_1,X_2}(x_1, t_1; x_2, t_2) dx_1 dx_2$$

where $h(\cdot,\cdot)$ is a function of two variables.

(cf) Note that $E\{h[X(t_1), X(t_2)]\}$, in general, is a function of both t_1 and t_2...

[1] Each can be calculated by: $E\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x,t) dx$ and $E\{X^2(t)\} = \int_{-\infty}^{\infty} x^2 f_X(x,t) dx$, respectively.

Autocorrelation function of a r.p. [2]

An important two-time average is the autocorrelation function, which is obtained by letting $h[X(t_1), X(t_2)] = X(t_1)X(t_2)$, so that:

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, t_1; x_2, t_2) dx_1 dx_2$$

Note:

(1) The autocorrelation function of a r.p. $X(t)$ at times t_1 and t_2 is the autocorrelation of two r.v.'s $X(t_1) = X_1$ and $X(t_2) = X_2$

(2) Likewise, the autocovariance is the covariance of the r.v.'s $X(t_1) = X_1$ and $X(t_2) = X_2$

Example 9.5

For the following r.p., find the statistical average mean and autocorrelation function.

$$X(t, \Theta) = A\cos(\omega_0 t + \Theta)$$

where A and ω_0 are constants, Θ is a r.v. $\ni$: $\Theta \sim U[0, 2\pi]$.

Solution:

Using the concept of the average of a function of a r.v., we have:

$$E[X(t;\theta)] = E[g(\theta)] = E[A\cos(\omega_0 t + \Theta)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta)\frac{d\theta}{2\pi} = 0$$

[2] This is called the ensemble-average autocorrelation function, compared to the time-average autocorrelation function to be discussed shortly...

Similarly, the autocorrelation function w/ $t_1 = t$ and $t_2 = t + \tau$ is as follows:

$$
\begin{aligned}
R_X(t, t+\tau) &= E[X(t)X(t+\tau)] \\
&= E\{A\cos(\omega_0 t + \Theta) A\cos[\omega_0(t+\tau) + \Theta]\} \\
&= \int_0^{2\pi} A\cos(\omega_0 t + \theta) A\cos[\omega_0(t+\tau) + \theta] \frac{d\theta}{2\pi} \\
&= \frac{A^2}{2} \int_0^{2\pi} [\cos(\omega_0 \tau) + \cos(2\omega_0 t + \omega_0 \tau + 2\theta)] \frac{d\theta}{2\pi} \\
&= \frac{A^2}{2} \left[\int_0^{2\pi} \cos(\omega_0 \tau) \frac{d\theta}{2\pi} + \int_0^{2\pi} \cos(2\omega_0 t + \omega_0 \tau + 2\theta) \frac{d\theta}{2\pi} \right] \\
&= \frac{A^2}{2} \cos(\omega_0 \tau) \\
&= R_X(\tau)
\end{aligned}
$$

Remark:

(1) Notice that the autocorrelation function depends only on $\tau = t_2 - t_1$.

(2) The mean squared value is $E[X^2(t)] = R_X(0) = A^2/2$.

9.3.3 Strictly Stationary Random Processes

Definition 9.6 Strict Sense Stationarity(sss)*:*
A r.p. $X(t)$ *is called* strict sense stationary *if it satisfies:*

$$f_X(x_1, x_2, \cdots, x_N; t_1, t_2, \cdots, t_N) = f_X(x_1, x_2, \cdots, x_N; t_2 - t_1, t_3 - t_1, \cdots, t_N - t_1)$$

Note:

(1) The time we pick for the origin makes no difference

(2) The 1st order pdf is independent of time, i.e. $f_X(x;t) = f_X(x;t+\Delta)\ \ \forall\, \Delta$.

(3) The mean and the mean square are independent of time.

(4) The autocorrelation function depends only on the time difference $\tau = t_2 - t_1$, i.e. $R_X(t_1, t_2) = R_X(t_2 - t_1) \triangleq R_X(\tau)$.

(5) Higher order time averages will be functions of the time difference as well.

9.3.4 Wide-Sense-Stationary Random Processes

Definition 9.7 Wide Sense Stationarity(wss): *A r.p. $X(t)$ is called* wide sense stationary *if it satisfies the following two conditions:*

$$E[X(t)] = constant$$

$$R_X(t_1, t_2) = R_X(\tau) \quad where \;\; \tau = t_2 - t_1$$

Note:

(1) Higher order ensemble averages may not be functions of time differences, compared to sss case.

(2) If a r.p. is sss, then it is wss, BUT not vice versa. [3]

(3) The $X(t)$ in example 9-5 is WSS. (why?)

9.3.5 Time Averages

For a stationary r.p., we can find averages over time, such that:

1. Time average mean:

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt$$

2. Time average mean square value:

$$\left\langle X^2(t) \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [X(t)]^2 dt$$

3. Time average autocorrelation function:

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)dt$$

Remark: Since the average is over time, the r.p. must be *stationary.*

[3] An exception is the case of a Gaussian r.p., where wss Gaussian r.p. is also sss.

9.3.6 Ergodic Processes

Definition 9.8 Ergodic Random Process [4]*:*
A r.p. $X(t)$ is called ergodic *if all statistical averages are equal to the their time average equivalents.*

Example 9.6

Consider a r.p. defined by:

$$X(t) = A, \quad -\infty < t < \infty$$

where A is a r.v..

Determine whether $X(t)$ is an ergodic r.p. or not.

Solution:

The ensemble average of $X(t)$ is:

$$E[X(t)] = E[A] = m_A = \text{constant}$$

The time average, on the other hand, is:

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A dt = \lim_{T\to\infty} \frac{1}{2T} A(2T) = A = \text{r.v.}$$

Therefore, the r.p. is NOT ergodic.

Note:
Notice that for a r.p. $X(t)$ to be ergodic, time averages should have a zero variance, i.e. should be constants, which means that each sample function must represents the whole r.p.!!!

[4] The ergodicity of a r.p. sometimes must be assumed like it or not, since the time averages are what we can implement practically.....

Example 9.7

For the following r.p., find the time average mean and autocorrelation function.

$$X(t,\Theta) = A\cos(\omega_0 t + \Theta)$$

where A and ω_0 are constants, Θ is a r.v. $\ni$: $\Theta \sim U[0, 2\pi]$.

Solution:

The time average mean is:

$$\langle X(t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \Theta)dt = 0 \quad \text{(why?)}$$

Similarly, the time average autocorrelation function is given as:

$$\begin{aligned}
R_X(\tau) &= \langle X(t)X(t+\tau)\rangle \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \theta)A\cos[\omega_0(t+\tau) + \theta]dt \\
&= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\cos(\omega_0\tau) + \cos(2\omega_0 t + \omega_0\tau + 2\theta)]dt \\
&= \frac{A^2}{2}\lim_{T\to\infty} \frac{1}{2T}\left\{\int_{-T}^{T} \cos(\omega_0\tau)dt + \int_{-T}^{T} \cos(2\omega_0 t + \omega_0\tau + 2\theta)dt\right\} \\
&= \frac{A^2}{2}\cos(\omega_0\tau)
\end{aligned}$$

Note:

(1) Notice that time and ensemble mean & autocorrelation functions are same.

(2) Is this r.p. then ergodic? We cannot say for sure [5]...

(3) The r.p.'s like this is called **ergodic in the wide sense**, where the time and ensemble averages are equal *up to and including second order*(i.e. the mean, variance, and autocorrelation function).

[5] Because it is necessary for all possible statistical averages to be equal to the corresponding time averages for a r.p. to be ergodic!

9.4 Autocorrelation Function Properties

One of the most important two-time average for a r.p. is the *autocorrelation function* defined by:

$$\begin{aligned} R_X(t_1, t_2) &= E\left[X(t_1)X(t_2)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, t_1; x_2, t_2) dx_1 dx_2 \end{aligned}$$

FACT: The autocorrelation function for a *stationary* r.p. depends only on difference of the two time instants, t_1 and t_2, at which the joint average is taken, i.e.:

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \triangleq R_X(\tau)$$

In addition to this.....

Properties of $R_X(\tau)$ for a stationary r.p. $X(t)$

1. The autocorrelation is maximum at the origin, i.e.:

$$|R_X(\tau)| \leq R_X(0)$$

2. The autocorrelation is an even function, i.e.:

$$R_X(-\tau) = R_X(\tau)$$

3. If $X(t)$ is ergodic, the mean square of $X(t)$ is the limiting value of the autocorrelation at infinity if it exists, i.e.

$$\lim_{\tau \to \infty} R_X(\tau) = \left\{E[X(t)]\right\}^2$$

4. The autocorrelation of a periodic r.p. is also periodic w/ the same period, i.e. if $X(t + T) = X(t)$ then:

$$R_X(\tau) = R_X(\tau + T)$$

5. The Fourier transform of the autocorrelation, referred to as *power spectral density* [6], is real and non-negative, i.e.:

$$S_X(f) \triangleq \mathcal{F}\{R_X(\tau)\} \geq 0$$

(cf) If, in addition, $X(t)$ is ergodic, the average power of the r.p. can be evaluated by either the time-autocorrealtion or the ensemble autocorrelation at $\tau = 0$:

$$\mathcal{R}_X(0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X^2(t)dt = \left\langle X^2(t) \right\rangle \overset{\text{(by ergodicity)}}{=} E[X^2(t)] = R_X(0)$$

Sketch of Proof:

1. Consider the following non-negative quantity,

$$[X(t) \pm X(t+\tau)]^2 \geq 0$$

Developing the LHS, we have:

$$X^2(t) \pm 2X(t)X(t+\tau) + X^2(t+\tau) \geq 0$$

Taking the mathematical expectation term by term, we get

$$E[X^2(t)] \pm 2E[X(t)X(t+\tau)] + E[X^2(t+\tau)] \geq 0$$

Since the r.p. $X(t)$ is assumed to be stationary, it can be expressed as:

$$R_X(0) \pm 2R_X(\tau) + R_X(0) \geq 0$$

Rearranging the inequality, we obtain

$$-R_X(0) \leq R_X(\tau) \leq R_X(0)$$

which is the property 1.

[6] This is called the *Wiener-Kinchine theorem.*

2. This can be proved via the change of variable as $t' = t + \tau$ in the definition of the statistical autocorrelation function as:

$$R_X(\tau) = E[X(t)X(t+\tau)] = E[X(t' - \tau)X(t')] = E[X(t')X(t' - \tau)] \triangleq R_X(-\tau)$$

3. We justify this heuristically by noting that as $|\tau| \to \infty$, the r.v.'s $X(t)$ and $X(t+\tau)$ become statistically independent if the process is not periodic, hence:

$$\lim_{\tau\to\infty} E[X(t)X(t+\tau)] = E[X(t)]E[X(t+\tau)] = \{E[X(t)]\}^2$$

4. Suppose $X(t) = X(t+T)$, then:

$$R_X(\tau + T) \triangleq E\left[X(t)X(t+\tau+T)\right] = E\left[X(t)X(t+\tau)\right] \triangleq R_X(\tau)$$

which means that $R_X(\tau)$ is also periodic w/ period T.

5. *self study*

(cf) The power spectral denisty(PSD) $S_X(f)$ represents the denisty of power in the r.p. with frequency, and when integrated over all frequency, we obtain the total average power of the process; for instance, consider the r.p. discussed in example 9-3, i.e., $X(t) = A\cos(\omega_0 t + \Theta)$ where $\Theta \sim U(0, 2\pi)$ of ehich the autocorrelation function is as follows:

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

The corresponding PSD is then:

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

Integrating over all f, we obtain the average power of the r.p. as:

$$\int_{-\infty}^{\infty} S_X(f)df = \frac{A^2}{4}\left[\int_{-\infty}^{\infty} \delta(f - f_0)df + \int_{-\infty}^{\infty} \delta(f + f_0)df\right] = \frac{A^2}{4}(1+1) = \frac{A^2}{2}\ \text{(watts)}$$

Note that this is the same result as computed via the autocorrelation function as:

$$R_X(0) = E[X^2(t)] = A^2/2\ \text{(watts)}$$

Example 9.8 *Self Study*

Example 9.9

Consider a r.p. with sample functions having the properties:

(a) The values taken on any time instant t_0 is either A or $-A$ w/ equal probability.

(b) The number k of switching instants in any interval T obeys a Poisson distribution.

$$P_k = \frac{(\alpha T)^k}{k!} e^{-\alpha T}, \quad k = 0, 1, 2, \ldots$$

Find the autocorrelation function and the PSD of the r.p..

Figure 9.5: Sample function from a random telegraph wave process.

Solution:

Let τ be any *positive* time interval,i.e. $\tau > 0$, then the statistical average autocorrealtion function is given as:

$$\begin{aligned} R_X(\tau) &= E[X(t)X(t+\tau)] \\ &= A^2 P[X(t) \text{and} X(t+\tau) \text{have same sign}] \\ &\quad +(-A^2) P[X(t) \text{and} X(t+\tau) \text{have opposite sign}] \\ &= A^2 P[\text{even number of switching instants in} (t, t+\tau)] \\ &\quad -A^2 P[\text{odd number of switching instants in} (t, t+\tau)] \end{aligned}$$

Using the Poisson distribution, the above $R_X(\tau)$ can be written as:

$$
\begin{aligned}
R_X(\tau) &= A^2 \sum_{k=0,\text{even}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau} - A^2 \sum_{k=1,\text{odd}}^{\infty} \frac{(\alpha\tau)^k}{k!} e^{-\alpha\tau} \\
&= A^2 e^{-\alpha\tau} \sum_{k=0}^{\infty} \frac{(-\alpha\tau)^k}{k!} \\
&= A^2 e^{-\alpha\tau} e^{-\alpha\tau} \\
&= A^2 e^{-2\alpha\tau}
\end{aligned}
$$

For negative τ, we use the fact that $R_X(-\tau) = R_X(\tau)$ and the overall autocorrelation function can be expressed as:

$$R_X(\tau) = A^2 e^{-2\alpha|\tau|}$$

Note:

(1) The average power of the r.p. is $P_{\text{tele.wave}} = R_X(0) = A^2$ (watts).

(2) The square of its mean is zero since $\{E[X(t)]\}^2 = \lim_{\tau\to\infty} R_X(\tau) = 0$, which implies that the mean is zero as well.

By taking the Fourier transform of the autocorrelation function, the PSD can be obtained as:

$$S_X(f) = \frac{A^2/\alpha}{1 + (\pi f/\alpha)^2}$$

Figure 9.6: The autocorrelation function and the PSD of a random telegraph signal with $\alpha = 0.5$.

Recall that when integrated over all frequencies, the PSD should give the average power of the r.p., which is checked below by carrying out the integral:

$$\begin{aligned}
P &= \int_{-\infty}^{\infty} \frac{A^2/\alpha}{1+(\pi f/\alpha)^2} df \\
&= 2\int_{0}^{\infty} \frac{A^2/\alpha}{1+(\pi f/\alpha)^2} df \\
&= \frac{2A^2}{\pi} \int_{0}^{\infty} \frac{du}{1+u^2} \\
&= \frac{2A^2}{\pi} \tan^{-1} u \Big|_0^{\infty} = A^2 \\
&= P_{\text{tele.wave}}
\end{aligned}$$

9.5 Cross-correlation and Covariance Functions

9.5.1 Joint Random Processes

It is often of interest to consider *two related r.p.'s* $\ni$: $X(t)$ and $Y(t)$

(e.g.) The input and the output r.p.'s of a system, such as a filter...

$\Longrightarrow$ We need the joint pdf of the values of each at an arbitrary number N of time instants:

$$f_{XY}(x_1, t_1; x_2, t_2; \ldots; x_N, t_N; y_1, t_1; y_2, t_2; \ldots; y_N, t_N)$$

Remarks:

1. If f_{XY} is independent of time origin, or a function of only the time differences $t_2 - t_1, t_3 - t_1, \ldots, t_N - t_1$, then the r.p. is called to be **jointly stationary in strict sense**.
2. If their joint moments of second order (e.g. variances, cross-correlation defined below as $E[X(t_1)Y(t_2)]$) are either constants or functions of time differences, the r.p. is said to be **jointly stationary in wide sense**.

9.5.2 Cross-Correlation Function

Definition 9.9 Cross-correlation Function*:*
Consider two wide-sense stationary r.p.'s $X(t)$ and $Y(t)$. Their cross-correlation function at $t_1 = t$ and $t_2 = t + \tau$ is defined as:

$$R_{XY}(\tau) = E[X(t_1)Y(t_2)] = E[X(t)Y(t+\tau)]$$

Remarks:

1. Two r.p.'s $X(t)$ and $Y(t)$ are called *orthogonal* if their cross-correlation function is zero:

$$R_{XY}(\tau) = 0 \quad \forall \ \tau$$

2. For two statistically independent r.p.'s $X(t)$ and $Y(t)$, $R_{XY}(\tau)$ can be expressed as:

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)]E[Y(t+\tau)] \quad \text{statistically independent r.p.'s} \\ &= E[X(t)]E[Y(t)] \quad \text{stationary and independent} \end{aligned}$$

   In this case, the r.p.'s are called to be *uncorrelated*

3. $R_{XY}(\tau)$ for two jointly stationary r.p.'s has the following properties:

   (a) $R_{XY}(-\tau) = R_{XY}(\tau)$
   (b) $|R_{XY}(\tau)| \leq [R_X(0)R_Y(0)]^{1/2}$
   (c) $|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0+)R_Y(0)]$

   proof: assignment

Example 9.10

Consider the following r.p. discussed in example 9-3 as an input to a filter that simply modifies the amplitude of each sample function to B and shifts the phase by a fixed amount ϕ. For the following r.p., find the time average mean and autocorrelation function.

$$X(t, \Theta) = A\cos(\omega_0 t + \Theta)$$

where A and ω_0 are constants, Θ is a r.v. $\ni$: $\Theta \sim U[0, 2\pi]$.

Find the input and output autocorrelation functions and the croos-correlation function b/w the input and the output.

Figure: The input and the output r.p.'s of the given system.

Solution:

Recall that, in example 9-5, the autocorrelation of the input $X(t)$ has been found to be:

$$R_X(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

Therefore, it is clear that the autocorrelation function of the output is

$$R_Y(\tau) = \frac{B^2}{2}\cos(\omega_0\tau)$$

The cross-correlation function is found from

$$\begin{aligned}
R_{XY}(\tau) &= E[X(t)Y(t+\tau)] = E\{A\cos(\omega_0 t + \theta)B\cos[\omega_0(t+\tau) + \theta + \phi]\} \\
&= \frac{AB}{2}E\{\cos(\omega_0\tau + \phi)B\cos[\omega_0(2t+\tau) + 2\theta + \phi]\} \\
&= \frac{AB}{2}\cos(\omega_0\tau + \phi)
\end{aligned}$$

A similar derivation shows that

$$R_{YX}(\tau) = \frac{AB}{2}\cos(\omega_0\tau - \phi)$$

Check: The properties of $R_{XY}(\tau)$ (assignment)

Example 9.11

Find the autocorrelation function of the r.p. $Z(t)$ defined below as the sum of two r.p.'s.

$$Z(t) \triangleq X(t) + Y(t)$$

Solution:

The desired autocorrelation function can be written as:

$$R_Z(\tau) = E\{[X(t) + Y(t)][X(t+\tau) + Y(t+\tau)]\}$$

Multiplying out the two sums and taking expectation term by term, we get

$$\begin{aligned} R_Z(\tau) &= E\{[X(t)X(t+\tau) + X(t)Y(t+\tau) + Y(t)X(t+\tau) + Y(t)Y(t+\tau)]\} \\ &= R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \\ &= R_X(\tau) + R_{XY}(\tau) + R_{XY}(-\tau) + R_Y(\tau) \end{aligned}$$

Note: If $X(t)$ and $Y(t)$ are *orthogonal*, then

(a) The autocorrelation becomes:

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau)$$

(b) The power in the sum of two r.p.'s is the sum of powers in separate r.p.'s, i.e.

$$P_Z = P_X + P_Y$$

9.5.3 Covariance Function

The covariance function of two jointly wide-sense stationary r.p.'s is defined as:

$$C_{XY}(\tau) = E\{[X(t) - E(X(t))][Y(t+\tau) - E(Y(t+\tau))]\}$$

This can be simplified to

$$\begin{aligned} C_{XY}(\tau) &= E[X(t)Y(t+\tau)] - E[X(t)]E[Y(t+\tau)] \\ &= R_{XY}(\tau) - \mu_X \mu_Y \end{aligned}$$

where $R_{XY}(\tau)$ is their cross-correlation function, $\mu_X = E[X(t)]$ and $\mu_Y = E[Y(t)]$.

NOTE: If $X(t)$ and $Y(t)$ are *uncorrelated*, then $C_{XY}(\tau) = 0 \;\; \forall \, \tau$.

9.6 Gaussian Random Processes

A r.p. $X(t)$ is called *Gaussian* if:

(1) The 1st order pdf at an arbitrary time t is Gaussian, i.e.:

$$f_X(x;t) = \frac{e^{-(x-\mu_X)^2/2\sigma_X^2}}{\sqrt{2\pi\sigma_X^2}}$$

where μ_X and σ_X^2 are functions of time t.

(2) The joint pdf at two arbitrary times, t_1 and t_2 is Gaussian, i.e.:

$$\begin{aligned} & f_{X_1X_2}(x_1, x_2; t_1, t_2)) \\ = & \frac{\exp\left[-\frac{(x_1-\mu_{X_1})^2/\sigma_{X_1}^2 - 2r(x_1-\mu_{X_1})(x_2-\mu_{X_2})/\sigma_{X_1}\sigma_{X_2} + (x_2-\mu_{X_2})^2/\sigma_{X_2}^2}{2(1-r^2)}\right]}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-r^2}} \end{aligned}$$

where μ_{X_1}, $\sigma_{X_1}^2$ and μ_{X_2}, $\sigma_{X_2}^2$ are functions of t_1 and t_2 respectively, and $r = r(t_1, t_2)$.

$$\vdots$$

(N) The joint pdf at N arbitrary time instants is the N-fold generalization of the above...

$\Longrightarrow$ We will mainly focus on this second order pdf here...

For a *stationary* Gaussian r.p. with zero mean, the 2nd order pdf becomes:

$$f(x_1, x_2; \tau) = \frac{\exp\left\{-\frac{x_1^2 - 2r(\tau)x_1x_2 + x_2^2}{2\sigma^2[1-r^2(\tau)]}\right\}}{2\pi\sigma^2\sqrt{1-r^2(\tau)}}$$

Note:

1. x_1 and x_2 denote the values of r.v.'s $X(t)$ at times $t = t_1$ and $t = t_2 = t + \tau$.
2. Since the r.p. is *stationary*, the joint pdf does not depend on both time instants t and $t + \tau$, but only on their difference τ.
3. The variances are time independent, i.e. $\sigma_{X_1}^2 = \sigma_{X_1}^2 \triangleq \sigma^2$, since the r.p. is *stationary.*
4. The function $r(\tau)$ is the normalized autocorrelation function (or covariance [7]), called the correlation coefficient function, given by

$$r(\tau) = \frac{R(\tau)}{\sigma^2} = \frac{R(\tau)}{R(0)} = \frac{E[X(t)X(t+\tau)]}{E[X^2(t)]}$$

Remark: *The two fold pdf of a stationary Gaussian r.p. $X(t)$ with zero mean can be COMPLETELY specified by the parameter σ^2 and the function $r(\tau)$!!!*

Useful properties of Gaussian r.p.:

(1) Any linear opeartion on a Gaussian r.p. produces another Gaussian r.p..

(2) Sum of two or more Gaussian r.p.'s, independent or not, is also a Gaussian r.p..

(3) The superposition of a large number of r.p.'s that are not Gaussian will tend to be a Gaussian r.p.. (dur to the central limit theorem...)

$\Longrightarrow$ **Gaussian r.p.'s are important models for analysis and design of systems subject to random variation such as random noise, measurement errors, and corrupted signals.....**

[7] This is because the means are assumed to be zero...

Example 9.12

Consider the integrator-thresholder in fig 9-7. The input is either one of the following signals with equal probability:

$$Y(t) = A + N(t),\ 0 \leq t \leq T$$

$$Y(t) = -A + N(t),\ 0 \leq t \leq T$$

where A and T are constants, and the noise $N(t)$ is a zero mean Gaussian r.p. with its autocorrelation function as [8]:

$$R(\tau) = \sigma^2 r(\tau) = \frac{N_0}{2}\delta(\tau)$$

The putput of the integrator at the end of T seconds is compared to a threshold of zero, and the decision on whether the signal component is A or $-A$ is made. Determine the followings:

(a) The probability of making an error given that A was really sent.

(b) The probability of making an error given that $-A$ was really sent.

(c) The average probability of making and error, and plot it as a function of A^2T/N_0 which correspond to the signal-to-noise ratio(SNR).

Figure 9.7: Integrator-threshold device for detecting a constant signal in Gaussian noise.

Solution:

The output of the integrator can be written as:

$$Z = \pm AT + N_I$$

where N_I is a Gaussian r.v. given by:

$$N_I = \int_0^T N(t)dt$$

[8]Such a r.p. is called *white* since all frequencies are present in equal power and therefore resembles white light in this respect...

Notice that N_I is Gaussian since it is the result of a linear operation on a Gaussian r.p..(recall!)

The pdf of N_I, therefore, can be completely specified by finding its mean and variance.....

The mean of N_I is given by

$$E[N_I] = E\left[\int_0^T N(t)dt\right] = \int_0^T E[N(t)]dt = 0$$

The variance of N_I is same as the second moment since the mean of N_I turned out to be zero [9], i.e.:

$$\begin{aligned}
\sigma_Z^2 &= E[N_I^2] = E\left\{\left[\int_0^T N(t)dt\right]^2\right\} = E\left[\int_0^T \int_0^T N(t)N(\zeta)dtd\zeta\right] \\
&= \int_0^T \int_0^T E[N(t)N(\zeta)]dtd\zeta = \int_0^T \int_0^T \frac{N_0}{2} r(\zeta - t)dtd\zeta \\
&= \int_0^T \int_0^T \frac{N_0}{2}\delta(\zeta - t)dtd\zeta = \int_0^T \frac{N_0}{2}dt = \frac{N_0 T}{2}
\end{aligned}$$

(a) The probability of an error given that A was really sent is then:

$$\begin{aligned}
P(E|\text{Apresent}) &= P(AT + N_I < 0) = P(N_I < -AT) \\
&= \int_{-\infty}^{-AT} \frac{e^{-z^2/2\sigma_Z^2}}{\sqrt{2\pi\sigma_Z^2}} = \int_{AT}^{\infty} \frac{e^{-z^2/2\sigma_Z^2}}{\sqrt{2\pi\sigma_Z^2}}
\end{aligned}$$

Make a change of variable as

$$u = \frac{z}{\sigma_Z}$$

Then the above error probability can be written as:

$$P(E|\text{Apresent}) = \int_{AT/\sigma_Z}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} dz = Q\left(\frac{AT}{\sigma_Z}\right)$$

(b) By symmetry, the probability of an error given $-A$ was sent is the same, i.e.:

$$P(E| - \text{Apresent}) = Q\left(\frac{AT}{\sigma_Z}\right)$$

[9] Also notice that the variances of Z and N_I are the same...

(c) The average error probability becomes:

$$
\begin{aligned}
P_E &= \frac{1}{2}P(E|A\text{present}) + \frac{1}{2}P(E|-A\text{present}) \\
&= Q\left(\frac{AT}{\sigma_Z}\right) = Q\left(\sqrt{\frac{2A^2T}{N_0}}\right)
\end{aligned}
$$

where the variance of Z is substituted as $\sigma_Z^2 = \frac{N_0 T}{2}$.

(cf) The quantity A^2T/N_0 is called the *signal-to-noise ratio*, and denoted as SNR for abbreviation.

Figure 9.8: Probability of error vs. SNR.

9.7 Discrete-Time Random Processes: The Autoregressive Process

Consider a discrete-time r.p. defined by the following difference equation:

$$X_{k+1} = \alpha X_k + (1-\alpha)N_k, \quad k = 0, 1, 2, \ldots$$

where α is a parameter in $[0,1]$, and N_k are iid [10] r.v.'s, which are assumed to be Gaussian with zero mean and variance of σ_n^2.

First order autoregressive r.p.

The above r.p. is called *1st order autoregressive r.p.* if X_{k+1} depends only on the immediate preceding sample X_k.

For several values of k, we have:

$$\begin{aligned} X_1 &= \alpha X_0 + (1-\alpha)N_0 = (1-\alpha)N_0 \\ X_2 &= \alpha X_1 + (1-\alpha)N_1 \\ X_3 &= \alpha X_2 + (1-\alpha)N_2 \\ X_4 &= \alpha X_3 + (1-\alpha)N_3 \end{aligned}$$

where the initial value of X_k at $k = -$ is assumed to be zero.

Substitution of each equation into the immediately following equation gives:

$$X_4 = (1-\alpha)(\alpha^3 N_0 + \alpha^2 N_1 + \alpha N_2 + N_3)$$

Using the fact that N_k's are iid and have zero mean, we find the variance of X_4 to be

$$\begin{aligned} \mathrm{Var}(X_4) &= (1-\alpha)^2[\alpha^6 \mathrm{Var}(N_0) + \alpha^4 \mathrm{Var}(N_1) + \alpha^2 \mathrm{Var}(N_2) + \alpha^0 \mathrm{Var}(N_3) \\ &= (1-\alpha)^2(1+\alpha^2+\alpha^4+\alpha^6)\sigma_n^2 \end{aligned}$$

Generalizing to arbitrary k from the above patter, we surmise that

$$\begin{aligned} \mathrm{Var}(X_k) &= (1-\alpha)^2(1+\alpha^2+\cdots+\alpha^{2(k-1)})\sigma_n^2 \\ &= (1-\alpha)^2\frac{1-\alpha^{2k}}{1-\alpha^2}\sigma_n^2 = \frac{1-\alpha}{1+\alpha}(1-\alpha^{2k})\sigma_n^2 \end{aligned}$$

which can be proved by the procedure of mathematical induction[11].

[10] independent, identically distributed

[11] Here, the summation formula for geometric series $\sum_{n=0}^{k-1} x^n = \frac{1-x^k}{1-x}$ has be use.

The process has a startup period that approaches to steady-state behavior as the index k increases...

$\Longrightarrow$ The variance of X_k versus k provides a good indication of the duration until approaching the steady state.

$\Longrightarrow$ The speed of this approach to steady state depends on α, i.e. the settling-out period is short for α close to 0, whereas for α close to 1 the settling-out period is longer...[figure 9-9($\alpha = 0.5$) and 9-10($\alpha = 0.95$)]

Figure 9.9: A sample function and the variance of and autoregressive X_k vs. k with $\alpha = 0.5$

Figure 9.10: A sample function and the variance of and autoregressive X_k vs. k with $\alpha = 0.95$

Remark: As k increases, the r.p. "forgets" its transient, and approaches to a stationary behavior.....

We now investigate the autocorrelation function of the 1st order autoregressive r.p. X_k, and we start with $R_X(0)$ as:

$$
\begin{aligned}
R_X(0) = E(X_k^2) &= E\{[\alpha X_k + (1-\alpha)N_k]^2\} \\
&= E[\alpha^2 X_k^2 + 2\alpha(1-\alpha)X_k N_k + (1-\alpha)^2 N_k^2] \\
&= \alpha^2 E(X_k^2) + (1-\alpha)^2\sigma_n^2 = \alpha^2 R_X(0) + (1-\alpha)^2\sigma_n^2
\end{aligned}
$$

where the fact that N_k is zero mean and independent of X_k has been used, i.e. $E[X_k N_k] = 0$.

Solving above w.r.t. $R_X(0)$, we obtain

$$R_X(0) = \frac{1-\alpha}{1+\alpha}\sigma_n^2$$

Now consider $R_X(1)$, which can be wriiten as:

$$
\begin{aligned}
R_X(1) &= E(X_k X_{k+1}) \\
&= E\{X_k[\alpha X_k + (1-\alpha)N_k]\} \\
&= \alpha E(X_k^[2) + (1-\alpha)EX_k N_k) \\
&= \alpha R_X(0)
\end{aligned}
$$

Repeating the procedure, we can show, in general, that:

$$R_X(M) = \alpha^{|M|} R_X(0) = \alpha^{|M|}\frac{1-\alpha}{1+\alpha}\sigma_n^2$$

Figure 9.11: Normalized $R_X(M)$ of an autoregressive r.p. for $\alpha = 0.5$ and $\alpha = 0.95$.

Recall: The autocorrelation function shows the interdependence b/w samples of the r.p..

$\Longrightarrow$ The process "forgets" about its past (or unable to predict its future) – more so as α becomes smaller.