Contents

Chapter 10

DISCRETE FOURIER TRANSFORM

10.1 Representation of DFT

Suppose we are given a *finite duration* [1] (N point) discrete-time signal $x[n] \ni$:

$$x[n] = 0 \quad \text{for } n < 0,\ n \geq N$$

e.g.

Figure 10.1: Finite duration discrete-time signal $x[n]$.

Following a similar procedure of deriving DTFT from DFS, we formulate the DFT pair for a finite duration sequence $x[n]$, i.e.

$\Longrightarrow$ **We analyze $x[n]$ by constructing a periodic $\tilde{x}[n]$, and taking only for** $0 \leq n \leq N-1$

[1]DFT is a practical tool for analyzing the frequency distribution of discrete-time signal, since we only can consider a finite duration sequence in real world!!!

Define a periodic sequence $\tilde{x}[n]$ as a repetition of $x[n]$ for $0 \le n \le N-1$, such that:

$$\tilde{x}[n] = x[n], \quad 0 \le n \le N-1$$

and

$$\tilde{x}[n] = \tilde{x}[n + m \cdot N]$$

Then, we can express $\tilde{x}[n]$ as a discrete Fourier series(DFS) pair as follows:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{D}_x(k) e^{j\frac{2\pi kn}{N}}$$

$$\tilde{D}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}}$$

The DFS coefficient $\tilde{D}_x(k)$ above can then be expressed as:

$$\begin{aligned} \tilde{D}_x(k) &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\ &\quad \text{(since } \tilde{x}[n] = x[n] \text{ for } 0 \le n \le N-1\text{)} \end{aligned}$$

Define a new function $X(k)$ of k as:

$$\begin{aligned} X(k) &\triangleq N \cdot \tilde{D}_x(k) \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \end{aligned} \tag{10.1}$$

which is called the discrete Fourier transform(DFT) of a finite duration discrete-time signal.

Note:

1. DFS coefficient $\tilde{D}_x(k)$ in terms of $X(k)$ is expressed as:

$$\tilde{D}_x(k) = \frac{1}{N} X(k)$$

2. Essentially, the DFT $X(k)$ of $x[n]$ is merely a scaled version of the DFS coefficient $\tilde{D}_x(k)$ of $\tilde{x}[n]$.

Then, from the DFS pair of $\tilde{x}[n]$, we have:

$$\begin{aligned}\tilde{x}[n] &= \sum_{k=0}^{N-1} \tilde{D}_x(k) e^{j\frac{2\pi kn}{N}} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}\end{aligned}$$

If we take $\tilde{x}[n]$ for only for $0 \le n \le N-1$, we get:

$$\tilde{x}[n] \equiv x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}, \quad 0 \le n \le N-1 \tag{10.2}$$

(10.1) and (10.2) are called the discrete Fourier transform (DFT) pair for a finite duration discrete-time signal $x[n]$:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad : \text{DFT}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}, \quad : \text{IDFT}$$

Remarks:

1. The DFT is, in general, complex. i.e.:

$$X(k) = \text{Re}[X(k)] + j\text{Im}[X(k)]$$

   where the real and imaginary parts for *real* discrete signal $x[n]$ are respectively expressed as follows:

$$\text{Re}[X(k)] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{2\pi kn}{N}\right)$$

$$\text{Im}[X(k)] = -\sum_{n=0}^{N-1} x[n] \sin\left(\frac{2\pi kn}{N}\right)$$

2. The DFT $X(k)$ of $x[n]$ is a sampled version of its DTFT [2] $X\left(e^{j\omega}\right)$, i.e.,

$$X(k) = X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}=k\omega_0}$$

   where $\omega_0 = \frac{2\pi}{N}$ is the sampling space.

   (cf.) Recall that $\tilde{D}_x(k)$ is a sampled version of $\frac{1}{N}X\left(e^{j\omega}\right)$, and since $X(k) = N \cdot \tilde{D}_x(k)$, it is clear that the DFT is in a close relationship to the DFS and the DTFT.

3. There exists a very fast algorithm called FFT(Fast Fourier Transform) to compute the DFT of discrete-time signals usually available at most computers and/or hardwares.

Figure 10.2: Block diagram of FFT

4. DFT is an essential(indispensable) practical tool for digitally processing signals using digital hardware/software. (reminder: we only deal with finite duration discrete-time signals in real world.)

[2] We can consider the finite duration sequence as a non-periodic sequence, and its DTFT is thus: $X\left(e^{j\omega}\right) \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$

10.2 Properties of DFT

Since the DFT is in close relation to the DFS and DTFT, its properties are also very similar to those of DFS ann DTFT.

Let

$$X(k) \triangleq \mathrm{DFT}_N[x[n]] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

where $x[n]$ is an N point finite duration discrete-time signal.

Then, some typical and important properties of the DFT are as follows:

(1) Periodicity:

$X(k)$ is periodic in k with period of N, i.e.

$$X(k) = X(k + m \cdot N) \qquad m : \text{integer}$$

proof:

$$\begin{aligned}
\mathrm{RHS} &\triangleq \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi(k+mN)n}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \cdot e^{-j\frac{2\pi mNn}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= X(k) = \mathrm{LHS}
\end{aligned}$$

(cf.) Since $X(k) \triangleq N \cdot \tilde{D}_x(k)$, and $\tilde{D}_x(k)$ is periodic, it is obvious that $X(k)$ should be periodic.

(2) Conjugate Symmetricity:

If $x[n]$ is a *real* discrete-time signal, then

$$X(-k) = X^*(k)$$

i.e.

$$\mathrm{Re}[X(k)] = \mathrm{Re}[X(-k)] \quad : \text{ even function of } k$$

$$\mathrm{Im}[X(k)] = -\mathrm{Im}[X(-k)] \quad : \text{ odd function of } k$$

proof:

$$\begin{aligned}
\text{LHS} = X(-k) &\triangleq \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi kn}{N}} \\
&= \left[\sum_{n=0}^{N-1} x^*[n]e^{-j\frac{2\pi kn}{N}}\right]^* \\
&= \left[\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}\right]^* \\
&\quad (\text{ since } x[n] \text{ is real}) \\
&= X^*(k) = \text{RHS}
\end{aligned}$$

OR

$$\begin{aligned}
\text{LHS} = X(-k) &\triangleq \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi(-k)n}{N}} \\
&= \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi kn}{N}} \\
&= \sum_{n=0}^{N-1} x[n]\left\{\cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} = X^*(k) &\triangleq \left[\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}\right]^* \\
&= \left[\sum_{n=0}^{N-1} x[n]\left\{\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right\}\right]^* \\
&= \sum_{n=0}^{N-1} x[n]\left\{\cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)\right\} \\
&\quad (\text{ since } x[n] \text{ is real})
\end{aligned}$$

Therefore,

$$\text{LHS} = \text{RHS}$$

(3) Combination of (1) and (2):

From (1) and (2), for *real* discrete-time signal $x[n]$, we have:

$$X^*(k) \overset{(2)}{=} X(-k) \overset{(1)}{=} X(N-k)$$

i.e.

$$\text{Re}[X(k)] = \text{Re}[X(N-k)] \quad : \text{symmetric about } k = \tfrac{N}{2}$$

$$\text{Im}[X(k)] = -\text{Im}[X(N-k)] \quad : \text{anti-symmetric about } k = \tfrac{N}{2}$$

e.g.

For $N = 6$,

Figure 10.3: Real and Imaginary parts of DFT for finite duration($N = 6$) discrete-time signal $x[n]$.

Note:

1. To maintain the periodicity and the conjugate symmetricity simulataneously, the imaginary part of DFT must be **zero** at $k = 0$, $k = N$, and $k = \frac{N}{2}$:

   proof:

   Let $X_I(k) \triangleq \text{Im}[X(k)]$, then

   $$\begin{cases} X_I(k) = X_I(k+N) & \text{:periodicity} \\ \\ X_I(k) = -X_I(N-k) & \text{:anti-symmetricity} \end{cases}$$

   Insert $k = 0$ in both equations, and then we get

   $$\begin{cases} X_I(0) = X_I(N) \\ \\ X_I(0) = -X_I(N) \end{cases}$$

   which means that

   $$X_I(0) = X_I(N) = 0$$

2. All the information that we need on the DFT $X(k)$ of N point discrete-time signal $x[n]$ is the values for half the period of $X(k)$, which is due to the periodicity and the conjugate symmetricity properties of DFT.

$\therefore$ **All we need for DFT $X(k)$ of N point sequence $x[n]$ is $X(k)$ for $k = 0, 1, 2, \ldots, [\frac{N}{2}]$ where $[m]$=largest integer $n \ni$: $n \leq m$**

Example 10.1

Find the DFT of the finite duration $(N = 8)$ discrete signal given below:

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 3 \\ \\ 0 & 4 \leq n \leq 7 \end{cases} \quad : \text{signal duration } N = 8$$

Figure 10.4: A 8 point discrete signal $x[n]$

Solution:

Re $[X(k)]$: symmetric$(\frac{N}{2})$ Im $[X(k)]$: anti-symmetric$(\frac{N}{2})$

Figure 10.5: Re $[X(k)]$ and Im $[X(k)]$ of 8 point discrete signal $x[n]$

Example 10.2

Find the DFT of a finite duration ($N = 4$) discrete cosine signal given below [3] :

$$x[n] = \cos\left(\frac{\pi n}{2}\right) \qquad \text{where } N = 4$$

Figure 10.6: A 4 point discrete cosine signal $x[n]$

Solution:

[3]This cosine sequence can be considered as the uniformly sampled result of a continuous cosine signal $\cos(t)$ with sampling period of $T_s = \frac{\pi}{2}$(sec): i.e. $\cos(t) \longrightarrow \cos(nT_s)$ with $T_s = \frac{\pi}{2}$.

Figure 10.7: DFT $X(k)$ of 4 point discrete cosine signal $x[n]$

Example 10.3

Find the DFT of a finite duration ($N = 4$) discrete sine signal given below [4] :

$$x[n] = \sin\left(\frac{\pi n}{2}\right) \qquad \text{where } N = 4$$

Figure 10.8: A 4 point discrete sine signal $x[n]$

Solution: Similary as in the previous example, we get

$$X(k) \triangleq \mathrm{DFT}_4\left[x[n]\right] = -2j\delta[k-1] + 2j\delta[k+1] \qquad : \text{pure imaginary!!!}$$

derivation: assignment

Figure 10.9: DFT $X(k)$ of 4 point discrete sine signal $x[n]$

[4]This sine sequence again can be considered as the uniformly sampled result of a continuous sine signal $\sin(t)$ with sampling period of $T_s = \frac{\pi}{2}$(sec): i.e. $\sin(t) \longrightarrow \sin(nT_s)$ with $T_s = \frac{\pi}{2}$.

Example 10.4

Write and run programs to compute the DFT's of above three examples, and compare the results with analytical solutions.

FACT:
In general, for N point sinusoidal discrete signals, we have:

$$\mathrm{DFT}_N\left[\cos\left(\frac{2\pi mn}{N}\right)\right] = \frac{N}{2}\left[\delta[k-m]+\delta[k+m]\right]$$

$$\mathrm{DFT}_N\left[\sin\left(\frac{2\pi mn}{N}\right)\right] = -j\frac{N}{2}\left[\delta[k-m]-\delta[k+m]\right]$$

where m is a fixed integer representing the number of cycles within N points.

Note: The above formula indicate that the only frequency component in these sinusoidal discrete signal is m times the fundamental frequency $\omega_0 = \frac{2\pi}{N}$ (radian), i.e. $m \cdot \omega_0 = m \cdot \frac{2\pi}{N}$.

SUMMARY OF TRANSFORMS:

	TIME	FREQUENCY
F.S.	$x(t)=\sum_{k=-\infty}^{\infty} C_x(k) e^{j \frac{2\pi kt}{T}}$ (i) continuous(t) (ii) periodic(T)	$C_x(k)=\frac{1}{T}\int_T x(t) e^{-j\frac{2\pi kt}{T}} dt$ (i) discrete(k) (ii) non-periodic
F.T.	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ (i) continuous(t) (ii) non-periodic	$X(\omega)=\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ (i) continuous(ω) (ii) non-periodic
DFT (DFS)	$x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}}$ (i) discrete(n) (ii) periodic(N)	$X(k)=\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$ (i) discrete(k) (ii) periodic(N)
DTFT	$x[n]=\frac{1}{2\pi}\int_0^{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$ (i) discrete(n) (ii) non-periodic	$X\left(e^{j\omega}\right)=\sum_{-\infty}^{\infty} x[n] e^{-j\omega n}$ (i) continuous(ω) (ii) periodic(2π)

NOTE:

<u>One domain</u>		<u>Other domain</u>
discrete	$\longleftrightarrow$	periodic
continuous	$\Longleftrightarrow$	non-periodic