Contents

Chapter 11

SAMPLING THEOREM

11.1 Background

We want to represent a continuous-time signal $x(t)$ by a discrete-time signal $x[n]$, which is the *sampled version* of the original continuous-time signal:

$$x(t) = \cos(t)$$

Figure 11.1: Sampled discrete-time signal $x[n]$ with two different sampling periods $T_s^{'} \ll T_s$.

Note:

1. T_s is called the sampling period, and $f_s = \frac{1}{T_s}$(Hz) or $\omega_s = 2\pi f_s$(rad/sec) is called the sampling rate or sampling frequency.

2. For $T_s = \frac{\pi}{2}$(sec), the number of samples for one period of $x(t)$ is $N = 4$, and for $T_s^{'} = \frac{\pi}{8}$(sec), it is $N^{'} = 16$ points.

3. The sampled discrete-time signal can be represented in two different ways, i.e.

$$x[n] = x(nT_s)$$

or

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

4. Notice that based on the samples signal $x[n]$, it is much easier to figure out that the original signal $x(t)$ was a cosine signal for the case of $T_s^{'} = \frac{\pi}{8}$ than the case of $T_s = \frac{\pi}{2}$.

QUESTION:
How high should the sampling rate($f_s \triangleq \frac{1}{T_s}$) be to faithfully represent the continuous-time signal $x(t)$ with the sampled discrete-time signal $x[n]$?

Answer:
As we can see in the above figures, **the higher** the sampling rate is, **the better** the representation is!

But, how high?

Remarks:

1. If f_s is too low, the sampled signal $x[n]$ may be ill defined.

2. If f_s gets higher, the sampled signal $x[n]$ might be well defined, but since the number of samples increases, the required hardware(such as memory) and/or the processing time gets longer, i.e. there $\ni$: trade-offs.

$\Longrightarrow$ We want to find out the minimum required value of f_s?

11.2 Analysis of Sampling

We consider the *uniform sampling* (i.e. the sampling period T_s is fixed!)

block diagram

$x(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad : \text{equally spaced impulse train}$$

$$x_s(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Figure 11.2: Sampling procedure with corresponding signals.

To analyze the sampling procedure in frequency domain, we first discuss the following fact:

FACT: *The Fourier transform(F.T.) of $p(t)$: train of impulses* [1]

1. Since $p(t)$ is periodic, it can be represented as a Fourier series(F.S.), i.e.

$$p(t) = \sum_{k=-\infty}^{\infty} C_p(k) e^{j\frac{2\pi kt}{T_s}}$$

where

$$C_p(k) = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t) e^{-j\frac{2\pi kt}{T_s}} dt = \frac{1}{T_s} = f_s, \quad \forall k$$

2. Therefore, we have:

$$p(t) = \sum_{k=-\infty}^{\infty} f_s \cdot e^{j\frac{2\pi kt}{T_s}}$$

and the Fourier transform of $p(t)$ is then

$$\begin{aligned}
P(\omega) &= \int_{-\infty}^{\infty} p(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} f_s \cdot e^{j\frac{2\pi kt}{T_s}} \right) e^{-j\omega t} dt \\
&= f_s \cdot \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\omega - \frac{2\pi k}{T_s})t} dt \\
&= f_s \cdot \sum_{k=-\infty}^{\infty} \mathcal{F}[1]\big|_{\omega \to \omega - k\omega_s} \qquad \text{where } \omega_s \triangleq \frac{2\pi}{T_s} \\
&= 2\pi f_s \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\
&= \omega_s \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)
\end{aligned}$$

(cf.) We can confirm the F.T. of $p(t)$ directly from the relation of F.T. and F.S. coefficient of periodic signals, which is:

$$2\pi C_x(k) = X\left(\frac{2\pi k}{T_s}\right)$$

where $x(t)$ is periodic with period of T_s.

Figure 11.3: Fourier transform of impulse train $p(t)$.

[1] Note that $p(t)$ is a singular function.

Now, from the block diagram,

$$x_s(t) = x(t) \cdot p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \tag{11.1}$$

Take the Fourier transform of (11.1), and we get:

$$\begin{aligned} X_s(\omega) &= \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right] \\ &= \frac{1}{2\pi}\left[X(\omega) * \omega_s \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right] \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} \{X(\omega) * \delta(\omega - k\omega_s)\} \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad \text{: by the sifting property of } \delta(\omega) \end{aligned} \tag{11.2}$$

Pictorially, if the F.T. of the original $x(t)$ has the following form:

Bandlimited signal

Figure 11.4: Fourier transform of the original $x(t)$.

Then, corresponding spectrum $X_s(\omega)$ of the sampled signal $x_s(t)$ will be as follows, which is a repetition of $X(\omega)$ with period of ω_s, scaled by $\frac{\omega_s}{2\pi}$, i.e.

LPF with BW=$\frac{\omega_s}{2}$, and Gain=$\frac{2\pi}{\omega_s}$

Figure 11.5: Fourier transform of the sampled signal $x_s(t)$: NO aliasing.

This is the case when the *lower edge* of the *second band (or sidelobe)* in $X_s(\omega)$ is greater than or equal to the *upper edge* of the *main band (or mainlobe)* in $X_s(\omega)$, i.e.

$$\omega_s - \omega_m \geq \omega_m \quad (\text{ or equivalently } \omega_s \geq 2\omega_m)$$

and we can recover $x(t)$ from $x_s(t)$ by way of a LPF with bandwidth of $\frac{\omega_s}{2}$, and gain of $\frac{2\pi}{\omega_s}$.

However, if the above condition is NOT satisfied, i.e. if:

$$\omega_s - \omega_m < \omega_m \quad (\text{ or equivalently } \omega_s < 2\omega_m)$$

then the spectrum of $X_s(\omega)$ will become as follows:

Figure 11.6: Fourier transform of the sampled signal $x_s(t)$: ALIASING.

It is obvious that $x(t)$ CANNOT be recovered from $x_s(t)$, which means that $x_s(t)$ does not faithfully represent the original $x(t)$, and this phenomenon is called the **ALIASING**!!!

From the above discussions, we have the following theorem:

Theorem 11.1 Sampling Theorem:

Let $x(t)$ be a *bandlimited* continuous-time signal with $X(\omega) = 0$ for $|\omega| > \omega_m$. Then, $x(t)$ is uniquely determined by its sampled discrete-time signal $x(nT_s)$, where $n = 0, \pm 1, \pm 2, \ldots$

IF the following condition is met:

$$\omega_s \geq 2\omega_m \qquad \text{(Nyquist Criterion)}$$

where $\omega_s = \frac{2\pi}{T_s}$ (rad/sec) is the sampling frequency.

Example 11.1

Usually, the audible signals such as speech, and music signals have typical frequency range from 2(Hz) to 20(KHz). Therefore, to record such audible signals in digital media like CD, and DAT, we need to sample the signal with at least $f_s = 40$(KHz) sampling rate.

(e.g.)
Audio CD's use a sampling rate of 44.1(KHz) for storage of the digital audio signals!!!

11.3 Interpolations

Intuition:

1. Recovering the continuous signal $x(t)$ from the sampled signal [2] $x_s(t)$.
2. Filling in with interpolated data between $x(nT_s)$ and $x((n+1)T_s)$ for $-\infty < n < \infty$.

11.3.1 Time domain interpolation

$$x_s(t) \quad \Longrightarrow \quad x(t)$$

$$\Downarrow$$

Figure 11.7: Time domain interpolation.

As mentioned in the previous section, when we discussed the sampling theorem, this can be accomplished by passing $x_s(t)$ through a low pass filter(LPF) with BW=$\frac{\omega_s}{2}$ and Gain=$\frac{2\pi}{\omega_s}$:

Figure 11.8: Interpolating LPF and spectrum $X_s(\omega)$.

(cf.) We assumed that the Nyquist criterion is met during the sampling process.

[2]Reminder: $x_s(t)$ can be expressed in another way $\ni$: $\{x(nT_s)\}_{n=-\infty}^{\infty}$

Let the transfer function of the LPF be $H(\omega)$, which should be as follows:

Figure 11.9: The transfer function of the LPF.

Then, the impulse response $h(t)$ of the LPF is:

$$
\begin{aligned}
h(t) = \mathcal{F}^{-1}\left[H(\omega)\right] &= \frac{1}{2\pi}\int_{-\frac{\omega s}{2}}^{\frac{\omega s}{2}} \frac{2\pi}{\omega_s} \cdot e^{j\omega t} d\omega \\
&= \frac{1}{2\pi} \cdot \frac{2\pi}{\omega_s}\left[\frac{e^{j\omega t}}{jt}\right]_{-\frac{\omega s}{2}}^{\frac{\omega s}{2}} \\
&= \frac{1}{\omega_s}\left[\frac{e^{j\frac{\omega s}{2}t} - e^{-j\frac{\omega s}{2}t}}{jt}\right] \\
&= \frac{1}{\omega_s} \cdot \frac{2j\sin\left(\frac{\omega_s}{2}t\right)}{jt} \\
&= \frac{\sin\left(\frac{\pi t}{T_s}\right)}{\frac{\pi t}{T_s}} \quad (\text{where } \omega_s = \tfrac{2\pi}{T_s}) \\
&\triangleq \text{sinc}\left(\frac{t}{T_s}\right)
\end{aligned}
$$

OR

Therefore, $x(t)$ can be considered as the output signal of the following LTI system!!!

Figure 11.10: The LPF as an LTI system with $h(t) = \text{sinc}\left(\frac{t}{T_s}\right)$.

i.e.

$$\begin{aligned}
x(t) &= x_s(t) * h(t) \\
&= \left[\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)\right] * \text{sinc}\left(\frac{t}{T_s}\right) \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \left[\delta(t - nT_s) * \text{sinc}\left(\frac{t}{T_s}\right)\right] \quad \text{(by the linearity of system)} \\
&= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \text{sinc}\left(\frac{t - nT_s}{T_s}\right) \quad \text{(by the time invariance of system)}
\end{aligned}$$

: This is why it is called the sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

Figure 11.11: Interpolated signal.

11.3.2 Frequency domain interpolation

$$x_N[n] \quad \Longrightarrow \quad x_{MN}[n]$$

where $x_N[n]$ is the sampled discrete signal from $x(t)$ using N points.

$$\Downarrow$$

Figure 11.12: Frequency domain interpolation, where $M = 2$.

Procedure:

1. Compute the N point DFT of $x_N[n]$, i.e. $X_N(k) = \text{DFT}_N\left[x_N[n]\right]$
2. Pad zeros between $\frac{N}{2} \leq k < \frac{3N}{2}$, and form a $2N$ point sequence of k, i.e. $X_{2N}(k)$.
3. Take the $2N$ point inverse DFT of $X_{2N}(k)$, i.e. $x_{2N}[n] = \text{DFT}_{2N}^{-1}\left[X_{2N}(k)\right]$
4. Multiply the resulting sequence by 2.

Then, we get the interpolated discrete-time signal $x_{2N}[n]$, which is twice as close to the original continuous-time signal $x(t)$.

Figure 11.13: Zero padding in frequency domain interpolation, where $M = 2$.

Analysis:

The zero padded $2N$ point sequence is as below:

$$X_{2N}(k) = \begin{cases} X_N(k), & 0 \leq k \leq \frac{N}{2} - 1 \\ 0, & \frac{N}{2} \leq k \leq \frac{3N}{2} - 1 \\ X_N(k-N) & \frac{3N}{2} \leq k \leq 2N - 1 \end{cases}$$

Take $2N$-point IDFT of $X_{2N}(k)$, then

$$\begin{aligned} x_{2N}[n] &\triangleq \frac{1}{2N} \sum_{k=0}^{2N-1} X_{2N}(k) e^{j\frac{2\pi kn}{2N}} \\ &= \frac{1}{2N} \left\{ \sum_{k=0}^{\frac{N}{2}-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} + \sum_{k=\frac{3N}{2}}^{2N-1} X_N(k-N) e^{j\frac{2\pi k(\frac{n}{2})}{N}} \right\} \\ &\quad (\text{ let } k' = k - N \text{ in the second term}) \\ &= \frac{1}{2N} \left\{ \sum_{k=0}^{\frac{N}{2}-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} + \sum_{k'=\frac{N}{2}}^{N-1} X_N(k') e^{j\frac{2\pi(k'+N)(\frac{n}{2})}{N}} \right\} \\ &= \frac{1}{2N} \left\{ \sum_{k=0}^{\frac{N}{2}-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} + \sum_{k=\frac{N}{2}}^{N-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} \cdot e^{j\pi n} \right\} \end{aligned}$$

(i) If $n = 2m$ (i.e. $e^{j\pi n} = 1$):

$$x_{2N}[n] = \frac{1}{2N} \sum_{k=0}^{N-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} = \frac{1}{2} x_N\left[\frac{n}{2}\right]$$

(ii) if $n = 2m + 1$ (i.e. $e^{j\pi n} = -1$):

$$x_{2N}[n] = \frac{1}{2N} \left\{ \sum_{k=0}^{\frac{N}{2}-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} - \sum_{k=\frac{N}{2}}^{N-1} X_N(k) e^{j\frac{2\pi k(\frac{n}{2})}{N}} \right\}$$

:which are the *intermediate* values of $x_N[n]$

$(e.g.)N = 5$

Figure 11.14: The original and the interpolated sequences $x_5[n]$ and $x_{10}[n]$.

Therefore, in order to recover the original amplitude as well, we have to multiply $x_{2N}[n]$ by factor of 2.

In general, if we pad zeros in $X_N[k]$ between $\frac{N}{2} \le k < (M - \frac{1}{2})N$ thus making it an MN point sequence $X_{MN}[k]$, and taking MN-point IDFT, we get:

$$x_{MN}[n] = \begin{cases} \frac{1}{M} x_N\left[\frac{n}{M}\right], & \text{if } n = m \cdot M \\ \text{intermediate values} & \text{if } n \neq m \cdot M \end{cases}$$

$\Longrightarrow$ We have to multiply M to $x_{MN}[n]$ to recover the original amplitude of $x_N[n]$!!!