Contents

Chapter 12

MISCELLANEOUS TOPICS

12.1 An Efficient Way of Computing Two DFT's of Real Sequences Simultaneously

As we mentioned before, there exists a fast algorithm for computing DFT, called the FFT(Fast Fourier Transform) algorithm, which usually requires *two input* arguments and gives *two outputs*, i.e.

Figure 12.1: DFT and equivalent FFT

In most of the times or cases, however, the input sequence $x[n]$ is a *real* discrete-time signal.

$\Longrightarrow \mathrm{Im}\,[x[n]] = 0 \quad \forall\, n = 0, 1, 2, \ldots, N-1$

$\Longrightarrow$ **Waste of resources!!!**

$\Longrightarrow$ Instead of putting zeros in $\mathrm{Im}\,[x[n]]$ array, we input another *real* sequence in it, by forming a complex sequence $g[n]$, i.e.

$$g[n] = x_1[n] + jx_2[n] \quad \text{:complex sequence}$$

Figure 12.2: FFT of two real discrete-time signals.

Objective:

We want to get $\mathrm{Re}\,[X_1(k)]$, $\mathrm{Im}\,[X_1(k)]$, $\mathrm{Re}\,[X_2(k)]$, and $\mathrm{Im}\,[X_2(k)]$ from $\mathrm{Re}\,[G(k)]$ and $\mathrm{Im}\,[G(k)]$.

Method:

Now,

$$\begin{aligned} X_1(k) &= \sum_{n=0}^{N-1} x_1[n] e^{-j\frac{2\pi kn}{N}} \\ &= \left\{ \sum_{n=0}^{N-1} x_1[n] \cos\left(\frac{2\pi kn}{N}\right) \right\} + j \left\{ -\sum_{n=0}^{N-1} x_1[n] \sin\left(\frac{2\pi kn}{N}\right) \right\} \\ &\triangleq \mathrm{Re}\,[X_1(k)] + j\mathrm{Im}\,[X_1(k)] \end{aligned} \tag{12.1}$$

Likewise, we have

$$\begin{aligned} X_2(k) &= \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{2\pi kn}{N}} \\ &= \left\{ \sum_{n=0}^{N-1} x_2[n] \cos\left(\frac{2\pi kn}{N}\right) \right\} + j \left\{ -\sum_{n=0}^{N-1} x_2[n] \sin\left(\frac{2\pi kn}{N}\right) \right\} \\ &\triangleq \mathrm{Re}\,[X_2(k)] + j\mathrm{Im}\,[X_2(k)] \end{aligned} \tag{12.2}$$

Let's take the DFT of $g[n]$,

$$\begin{aligned} G(k) &= \sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi kn}{N}} \\ &= \sum_{n=0}^{N-1} \{x_1[n] + jx_2[n]\} \cdot \left\{\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right\} \\ &= \sum_{n=0}^{N-1} \left\{x_1[n]\cos\left(\frac{2\pi kn}{N}\right) + x_2[n]\sin\left(\frac{2\pi kn}{N}\right)\right\} \\ &\quad +j\sum_{n=0}^{N-1} \left\{-x_1[n]\sin\left(\frac{2\pi kn}{N}\right) + x_2[n]\cos\left(\frac{2\pi kn}{N}\right)\right\} \\ &\triangleq \mathrm{Re}\left[G(k)\right] + j\mathrm{Im}\left[G(k)\right] \end{aligned} \tag{12.3}$$

From (12.1), (12.2), and (12.3), we have the following simultaneous equations:

$$\mathrm{Re}\left[G(k)\right] = \mathrm{Re}\left[X_1(k)\right] - \mathrm{Im}\left[X_2(k)\right] \tag{12.4}$$

$$\mathrm{Im}\left[G(k)\right] = \mathrm{Im}\left[X_1(k)\right] + \mathrm{Re}\left[X_2(k)\right] \tag{12.5}$$

Since we have four unknowns with two equations, we must find other two equations to solve:

Notice that:

(i) $x_1[n]$ and $x_2[n]$ are *real*.

(ii) $G(k)$ is periodic in k with period of N.

Therefore, from the properties of DFT of real discrete signals, such that the real part and the imaginary part of DFT are "symmetric" and "anti-symmetric", respectively, we have:

$$X_1(-k) = X_1^*(k)$$

$$X_2(-k) = X_2^*(k)$$

$$G(-k) = G(N-k)$$

Plugging $-k$ in place of k in (12.4) and (12.5), we get:

$$\mathrm{Re}\left[G(-k)\right] = \mathrm{Re}\left[G(N-k)\right] = \mathrm{Re}\left[X_1(k)\right] + \mathrm{Im}\left[X_2(k)\right] \tag{12.6}$$

$$\mathrm{Im}\left[G(-k)\right] = \mathrm{Im}\left[G(N-k)\right] = -\mathrm{Im}\left[X_1(k)\right] + \mathrm{Re}\left[X_2(k)\right] \tag{12.7}$$

Solving (12.4), (12.5), (12.6), and (12.7) simultaneously, we get

$$\left\{ \begin{array}{l} \mathrm{Re}\left[X_1(k)\right] = \frac{1}{2}\left\{\mathrm{Re}\left[G(k)\right] + \mathrm{Re}\left[G(N-k)\right]\right\} \\ \mathrm{Im}\left[X_1(k)\right] = \frac{1}{2}\left\{\mathrm{Im}\left[G(k)\right] - \mathrm{Im}\left[G(N-k)\right]\right\} \\ \mathrm{Re}\left[X_2(k)\right] = \frac{1}{2}\left\{\mathrm{Im}\left[G(k)\right] + \mathrm{Im}\left[G(N-k)\right]\right\} \\ \mathrm{Im}\left[X_2(k)\right] = \frac{1}{2}\left\{-\mathrm{Re}\left[G(k)\right] + \mathrm{Re}\left[G(N-k)\right]\right\} \end{array} \right\}$$

where $k = 0, 1, 2, \ldots, N-1$ (one period).

i.e.

Figure 12.3: Algorithm of computing DFT's of two real discrete-time signals.

$\Longrightarrow$ **We compute two DFT's $X_1(k)$ and $X_2(k)$ simultaneously, using FFT routine only once!!!**

Assignment:
Write a computer program to implemet the above algorithm, and check the result with those of individual DFT's.