Contents

Chapter 2

LINEAR TIME INVARIANT(LTI) SYSTEMS

2.1 Input/output Relation in LTI system: Convolution integral

$h(t)$ is the internal function representing the system characteristics

Figure 2.1: LTI system

FACT:

The input/output signals of the LTI system are related by a convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \triangleq x(t) * h(t) \tag{2.1}$$

where $h(t)$ is the *impulse response* of the system $T[\cdot]$.

System impulse response: $h(t)$

Definition 2.1 The impulse response $h(t)$ of an LTI system is defined as the output signal when the input signal is the *unit impulse function* $\delta(t)$, i.e.,

$$h(t) \triangleq T[\delta(t)]$$

Figure 2.2: Concept of impulse response for an LTI system

Unit impulse function(Dirac delta function): $\delta(t)$

Definition 2.2 The unit impulse function is defined as follows:

$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t)$$

where the *unit pulse* $\delta_\Delta(t)$ is

$$\delta_\Delta(t) \triangleq \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

Figure 2.3: Concept of unit impulse function

Then, we can re-define the unit impulse function $\delta(t)$ using following two conditions:

(a) magnitude:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{elsewhere} \end{cases}$$

(b) area:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \text{(unit area)}$$

Figure 2.4: Representation of unit impulse function

Check: validity of the definition of $h(t)$ using (2.1)

$$y(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = h(t)\int_{-\infty}^{\infty} \delta(\tau)d\tau = h(t)$$

which is the impulse response of the LTI system.

Figure 2.5: Validity of impulse response

Brief Derivation of (2.1)

Any continuous-time signal $x(t)$ can be represented as follows:

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta$$

where

$$\delta_\Delta(t - k\Delta) = \begin{cases} \frac{1}{\Delta}, & k\Delta \leq t < (k+1)\Delta \\ \\ 0, & \text{otherwise} \end{cases}$$

Figure 2.6: Approximated representation of continuous $x(t)$

The output signal $y(t)$ is expressed as:

$$\begin{aligned} y(t) &= T[x(t)] \\ &= T\left[\lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta\right] \end{aligned} \quad (2.2)$$

Here, let $h_\Delta(t) \triangleq T[\delta_\Delta(t)]$, then as $\Delta \to 0$, we have:

(1) $\lim_{\Delta \to 0} h_\Delta(t) = T[\lim_{\Delta \to 0} \delta_\Delta(t)] = T[\delta(t)] = h(t)$

(2) $k\Delta \longrightarrow \tau$ (continuous variable)

(3) $\Delta \longrightarrow d\tau$ (infinitesimal increment)

(4) $\lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$

Using the above facts, and due to the *linearity* and *time-invariance* properties of an LTI system, the output signal $y(t)$ of an LTI system (2.2) becomes:

$$\begin{aligned} y(t) &= T[\lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta] \\ &= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)T[\delta_\Delta(t - k\Delta)] \cdot \Delta \quad (\textbf{linearity}) \\ &= \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta) \cdot \Delta \quad (\textbf{time-invariance}) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\ &\triangleq x(t) * h(t) \end{aligned} \quad (2.3)$$

Another expression of the convolution

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned} \tag{2.4}$$

Let $t - \tau = t^{'}$ (change of variable from τ to $t^{'}$), then

(1) $\tau = t - t^{'}$

(2) $t^{'} \to \infty$ as $\tau \to -\infty$

(3) $t^{'} \to -\infty$ as $\tau \to \infty$

(4) $d\tau = -dt^{'}$

And (2.4) becomes:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{\infty}^{-\infty} x(t-t^{'})h(t^{'})(-dt^{'}) \\ &= \int_{-\infty}^{\infty} h(t^{'})x(t-t^{'})dt^{'} \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &\triangleq h(t) * x(t) \end{aligned} \tag{2.5}$$

Therefore, the output $y(t)$ of an LTI system can be obtained by:

$$y(t) = h(t) * x(t) \overset{OR}{=} x(t) * h(t)$$

Note:
The choice between (2.3) and (2.5) to compute $y(t)$ is entirely depending on the easiness of calculation w.r.t. the associated $x(t)$ and $h(t)$!!!

Example 2.1

Find the output signal $y(t)$ of an LTI system, when the input and the impulse response of the system are given repectively as follows:

$$x(t) = \sin(t)$$

$$h(t) = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & \text{otherwise} \end{cases}$$

2.2 Typical Causal LTI System

FACT:

If the impulse response of an LTI system satisfies:

$$h(t) = 0, \qquad \text{for } \ t < 0$$

Then, the system is a *causal* system.

Proof:

Figure 2.7: Input and output of a causal LTI system

$$\begin{aligned} y(t) = x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\mathbf{t}} x(\tau)h(t-\tau)d\tau \quad \forall t \end{aligned}$$

Note:
Notice that the output $y(t)$ depends only on $x(\tau)$ for $\tau \leq t$, and thus the system is *causal*!!!

Example 2.2

Analysis of an LTI system(time domain)

Figure 2.8: LTI system

For an LTI system with the input and the impulse response given below, find the output.

$$x(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} e^{-bt}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $a > 0$ and $b > 0$.

Solution:

Figure 2.9: Input, impulse response, and convolution integral

We have to compute the convolution integral between $h(t)$ and $x(t)$ as:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned}$$

.

Figure 2.10: Output signal $y(t)$

Note:

Comparing the input $x(t)$ and the output $y(t)$, we can notice that the system operates as a LPF(Low Pass Filter)!!!

Assignment: Try $y(t) = x(t) * h(t)$, and see if you get the same result!

Example 2.3

Analysis of an LTI system(time domain)

Repeat the above example given the input and the impulse response as follows:

$$x(t) = \begin{cases} 1, & 0 \le t \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$h(t) = \begin{cases} 2, & 1 \le t \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Solution:

Figure 2.11: Input, impulse response, and convolution integral

We now try to compute the convolution integral between $x(t)$ and $h(t)$ as:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \end{aligned}$$

Figure 2.12: Output signal $y(t)$

Comparing the input $x(t)$ and the output $y(t)$, we can notice that the system as well operates as a LPF(Low Pass Filter) as in the previous example!!!

Assignment: Try $y(t) = h(t) * x(t)$, and see if you get the same result!

Assignment: Which one ($y(t) = h(t) * x(t)$ or $y(t) = x(t) * h(t)$) is easier to compute for you in example 2.1?