Contents

Chapter 3

FOURIER SERIES

3.1 Concept of Fourier Series

Basic Idea

Given a periodic signal $x(t)$ with fundamental period of T_0, i.e.:

$$x(t) = x(t + T_0), \quad \forall\, t$$

Let's define the fundamental frequency of $x(t)$ as:

$$\omega_0 = \frac{2\pi}{T_0}\text{(rad/sec): angular frequency}^1$$

Then, we have the following facts:

1. $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ are periodic with period T_0
2. $\{\cos(n\omega_0 t)\}_{n=1}^{\infty}$ and $\{\sin(n\omega_0 t)\}_{n=1}^{\infty}$ are periodic with period T_0 (cf. $\frac{T_0}{n} < T_0$)
3. Linear combination of periodic signals(T_0) is also periodic with period of T_0, i.e.

$$\sum_{n=1}^{\infty} \{a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\}$$

[1]Corresponding cyclic frequency is $f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$(Hz)

Therefore, considering a d.c. component d_0, any periodic(T_0) signal can be represented by a linear combination of *harmonically related* sine and cosine functions:

$$\begin{aligned} x(t) &= d_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T_0} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{T_0} t\right) \\ &= d_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\omega_0 t\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\omega_0 t\right) \qquad (3.1) \end{aligned}$$

where $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and d_0 are to be determined depending on the specific $x(t)$.

Physical meaning:
d_0, a_n, and b_n represent the *magnitude(or contribution)* of each harmonic frequency component *d.c.*, $\cos(n\omega_0 t)$, and $\sin(n\omega_0 t)$ respectively in $x(t)$!!!

Example 3.1

Representation of periodic signal $x(t)$, whose period is $T_0 = 2\pi$(sec).

$$x(t) = \sin(t) + \sin(2t)$$

(cf.) The Fourier Series coefficients of $x(t)$ in the above example are $b_1 = b_2 = 1$, and all other coefficients are zero!

3.2 Trigonometric Representation of Fourier Series

Consider a *periodic* signal $x(t)$, and suppose it satisfies the *Dirichlet conditions*, i.e.

1. $x(t) = x(t + n \cdot T_0)$, $\quad n$: integer
2. Dirichlet Conditions:
   (a) $x(t)$ has a *finite* number of *finite* maxima and minima within the interval T_0.
   (b) $x(t)$ has a *finite* number of *finite* discontinuities within T_0.
   (c) $x(t)$ is *absolutely integrable* over T_0,i.e.

$$\int_{T_0} |x(t)| dt < \infty$$

Figure 3.1: A periodic signal satisfying Dirichlet conditions.

Then, $x(t)$ can be expressed as a linear combination of harmonically related sine and cosine functions:

$$\begin{aligned} x(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{2\pi nt}{T_0}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{2\pi nt}{T_0}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(n\omega_0 t\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(n\omega_0 t\right) \end{aligned} \tag{3.2}$$

where $\omega_0 = \frac{2\pi}{T_0}$ is the *fundamental frequency* of $x(t)$.

Corresponding F.S. coefficients $\{a_n\}_{n=\mathbf{0}}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are given as follows:

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos\left(\frac{2\pi nt}{T_0}\right)dt = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)dt \quad \text{:cosine of } n\text{-th harmonic}$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin\left(\frac{2\pi nt}{T_0}\right)dt = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)dt \quad \text{:sine of } n\text{-th harmonic}$$

Assume $T_0 = 2\pi$(sec) from now on WLOG[2], then $\omega_0 = 1$(rad/sec) and the Fourier series can simply be put as follows:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(nt)dt \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(nt)dt \end{aligned}$$

[2]WLOG: Without Loss Of Generality

Example 3.2

Express $x(t) = \cos(t)$ in a Fourier series.

Solution:

Example 3.3

(1) $x(t) = \cos^2(t)$

(2) $x(t) = \cos\{\tan(t)\}$

(3) $x(t) = \delta(t), \quad -\frac{\pi}{2} < t \leq \frac{\pi}{2}$, and $x(t) = x(t + 2\pi)$

3.3 Derivation of Fourier Series Coefficients

There exist two approaches to derive the trigonometric F.S. coefficients:

1. MSE (Mean Squared Error) minimization
2. Concept of vector based orthonormal basis for signal space

1. MSE minimization: (staightforward, but tedius to do)

First, we define the mean squared error(MSE) of the Fourier series representation of a continuous periodic signal $x(t)$ as:

$$\text{MSE} = \frac{1}{2\pi}\int_0^{2\pi} e^2(t)dt \qquad : \text{ we assumed } \; T_0 = 2\pi$$

where the error signal $e(t)$ is $e(t) \triangleq x(t) - x_F(t)$, and $x(t)$ is the original signal whereas $x_F(t)$ is the F.S. representation of $x(t)$.

Objective:
We want to find the F.S. coefficients $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, which minimizes the MSE. (i.e.: we want to have $x_F(t)$ be as close as to $x(t)$)

To achieve our objective, we first compute $e^2(t)$, where

$$\begin{aligned} e(t) &\triangleq x(t) - x_F(t) \\ &= x(t) - \left\{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)\right\} \end{aligned}$$

Then,

$$
\begin{aligned}
e^2(t) &= x^2(t) + \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \right\}^2 \\
&\quad -2x(t) \left\{ \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \right\} \\
&= x^2(t) + \left\{ \frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 \cos^2(nt) + \sum_{p,q=1}^{\infty} \sum_{p \neq q}^{\infty} a_p a_q \cos(pt)\cos(qt) \right. \\
&\quad + \sum_{n=1}^{\infty} b_n^2 \sin^2(nt) + \sum_{r,s=1}^{\infty} \sum_{r \neq s}^{\infty} b_r b_s \sin(rt)\sin(st) \\
&\quad + a_0 \sum_{n=1}^{\infty} a_n \cos(nt) + a_0 \sum_{n=1}^{\infty} b_n \sin(nt) \\
&\quad \left. + 2 \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} a_\alpha b_\beta \cos(\alpha t)\sin(\beta t) \right\} \\
&\quad - a_0 x(t) - 2x(t) \sum_{n=1}^{\infty} a_n \cos(nt) - 2x(t) \sum_{n=1}^{\infty} b_n \sin(nt) \qquad (3.3)
\end{aligned}
$$

We now take a look at MSE $= \frac{1}{2\pi}\int_0^{2\pi} e^2(t)dt$ term by term using (3.3):

(1) $\frac{1}{2\pi}\int_0^{2\pi} \cos^2(nt)dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left[1+\cos(2nt)\right]dt = \frac{1}{2\pi}\cdot\left(\frac{1}{2}2\pi\right) = \frac{1}{2}$

(2) $\frac{1}{2\pi}\int_0^{2\pi} \sin^2(nt)dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\left[1-\cos(2nt)\right]dt = \frac{1}{2\pi}\cdot\left(\frac{1}{2}2\pi\right) = \frac{1}{2}$

(3) $\frac{1}{2\pi}\int_0^{2\pi} \cos(nt)dt = 0$

(4) $\frac{1}{2\pi}\int_0^{2\pi} \sin(nt)dt = 0$

(5) $\frac{1}{2\pi}\int_0^{2\pi} \cos(pt)\cos(qt)dt = 0 \quad p \neq q$

(6) $\frac{1}{2\pi}\int_0^{2\pi} \sin(rt)\sin(st)dt = 0 \quad r \neq s$

(7) $\frac{1}{2\pi}\int_0^{2\pi} \cos(\alpha t)\sin(\beta t)dt = 0 \quad \forall \alpha, \beta$

For notational convenience, let's compute 2πMSE rather than MSE, then from the above calculation results, we have:

$$
\begin{aligned}
2\pi\text{MSE} &= \int_0^{2\pi} e^2(t)dt \\
&= \int_0^{2\pi} x^2(t)dt + \int_0^{2\pi} \frac{a_0^2}{4}dt + \pi\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) - a_0\int_0^{2\pi} x(t)dt \\
&\quad -2\sum_{n=1}^{\infty} a_n \int_0^{2\pi} x(t)\cos(nt)dt - 2\sum_{n=1}^{\infty} b_n \int_0^{2\pi} x(t)\sin(nt)dt \qquad (3.4)
\end{aligned}
$$

To get a_k's and b_k's which minimize the MSE, we differentiate (3.4) with respect to a_k, b_k, and put to zero:

1. a_0

$$\begin{aligned}\frac{\partial(2\pi \text{MSE})}{\partial a_0} &= \frac{\partial}{\partial a_0}\left\{\frac{a_0^2}{4}2\pi - a_0\int_0^{2\pi} x(t)dt\right\} \\ &= \pi a_0 - \int_0^{2\pi} x(t)dt \\ &= 0\end{aligned}$$

$$\Longrightarrow \quad a_0 = \frac{1}{\pi}\int_0^{2\pi} x(t)dt \quad \text{(twice the d.c. component)}$$

2. a_k

$$\begin{aligned}\frac{\partial(2\pi \text{MSE})}{\partial a_k} &= \frac{\partial}{\partial a_k}\left\{\pi\sum_{n=1}^{\infty} a_n^2 - 2\sum_{n=1}^{\infty} a_n \cdot \int_0^{2\pi} x(t)\cos(nt)dt\right\} \\ &= 2\pi a_k - 2\int_0^{2\pi} x(t)\cos(kt)dt \\ &= 2\left\{\pi a_k - \int_0^{2\pi} x(t)\cos(kt)dt\right\} \\ &= 0\end{aligned}$$

$$\Longrightarrow \quad a_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(kt)dt \qquad k = 1, 2, 3, \cdots$$

(cf.) Notice that the above formula for a_k is valid for $k = \mathbf{0}, 1, 2, 3, \cdots$.

3. b_k

$$\begin{aligned}\frac{\partial(2\pi \text{MSE})}{\partial b_k} &= \frac{\partial}{\partial b_k}\left\{\pi\sum_{n=1}^{\infty} b_n^2 - 2\sum_{n=1}^{\infty} b_n \cdot \int_0^{2\pi} x(t)\sin(nt)dt\right\} \\ &= 2\pi b_k - 2\int_0^{2\pi} x(t)\sin(kt)dt \\ &= 2\left\{\pi b_k - \int_0^{2\pi} x(t)\sin(kt)dt\right\} \\ &= 0\end{aligned}$$

$$\Longrightarrow \quad b_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(kt)dt \qquad k = 1, 2, 3, \cdots$$

Note:
To guarantee that above a_k, and b_k provide the *minimum* value for MSE, we have to show that MSE is a convex function at those points, i.e.

1. $\frac{\partial^2(2\pi\text{MSE})}{\partial a_0^2} = \pi > 0$

2. $\frac{\partial^2(2\pi\text{MSE})}{\partial a_k^2} = 2\pi > 0$

3. $\frac{\partial^2(2\pi\text{MSE})}{\partial b_k^2} = 2\pi > 0$

(cf.) For your reference, here are some trigonometric identities:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$$

$$\sin(\alpha)\sin(\beta) = -\frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$$

$$\cos(\alpha)\sin(\beta) = \frac{1}{2}\{\cos(\alpha+\beta) - \cos(\alpha-\beta)\}$$

2. Orthonornal basis for a signal space: [3]

Prerequisites and reviews:

We first review some of the basic concepts of vector spaces, and link them to the concept of signal space.

(1) Inner Product:

Define an inner product of two signals in a signal space $\mathbf{S}$ as:

$$< p(t), q(t) > \triangleq \frac{1}{\pi} \int_0^{2\pi} p(t) \cdot q(t) dt \tag{3.5}$$

where $p(t) \in \mathbf{S}$ and $q(t) \in \mathbf{S}$.

Note: Requirements for an inner product definition:

(i) Linearity: $< \alpha x_1 + \beta x_2, y >= \alpha < x_1, y > + \beta < x_2, y >$

(ii) Symmetry: $< x, y >=< y, x >$

(iii) Non-degeneray: $||x||^2 \triangleq < x, x > \geq 0$, and $||x||^2 = 0$ iff $x = 0$

Assignment: Check that (3.5) satisfies the above conditions (i) to (iii).

(2) Independence:

Signals $\{f_n(t)\}_{n=1}^{\infty} \in \mathbf{S}$ are called *independent* if the following condition is met:

$$\sum_{n=1}^{\infty} a_n \cdot f_n(t) = 0 \quad \text{iff} \quad a_n = 0 \ \ \forall n = 1, 2, \cdots$$

[3] We apply the concept of the vector space to a signal space

(3) Orthogonality:

Two signals $f_n(t) \in \mathbf{S}$ and $f_m(t) \in \mathbf{S}$ are called *orthogonal* if they satisfiy the follwoing condition:

$$< f_n(t), f_m(t) >= a_{n,m}\delta_{nm}$$

where δ_{nm} is called the Kronecker delta, and defined as follows:

$$\delta_{nm} \triangleq \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}$$

(4) Orthomormality:

Two signals $f_n(t) \in \mathbf{S}$ and $f_m(t) \in \mathbf{S}$ are called *orthonormal* if they satisfiy the following condition:

$$< f_n(t), f_m(t) >= \delta_{nm}$$

(5) Basis:

Basis of a signal space $\mathbf{S}$ is the minimum set of *independent* signals $\{f_n(t)\}_{n=1}^{N}$ such that any signal $x(t)$ in $\mathbf{S}$ can be represented by a linear combination of $\{f_n(t)\}_{n=1}^{N}$, i.e.

$$x(t) = \sum_{n=1}^{N} a_n \cdot f_n(t)$$

(cf.) Orthonormal basis: basis composed of orthonormal signals.

We now consider a signal space which is composed of periodic signals that can be represented by the Fourier Series, denoted as **F**:

$$\mathbf{F} = \left\{ x(t) | x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \right\} \quad \text{period} = 2\pi \tag{3.6}$$

Notice that:

1. Signals in **F** are linear combinations of 1, $\{\cos(nt)\}_{n=1}^{\infty}$, and $\{\sin(nt)\}_{n=1}^{\infty}$
2. 1, $\{\cos(nt)\}_{n=1}^{\infty}$, and $\{\sin(nt)\}_{n=1}^{\infty}$ are independent

And thus 1, $\{\cos(nt)\}_{n=1}^{\infty}$, and $\{\sin(nt)\}_{n=1}^{\infty}$ could be a BASIS for the signal space **F**.

Now, check the following facts:

(i) $< \sin(nt), \sin(mt) >= \delta_{nm}$

(ii) $< \cos(nt), \cos(mt) >= \delta_{nm}$

(iii) $< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >= 1$

(iv) $< \sin(nt), \cos(mt) >= 0, \quad \forall n, m$

(v) $< \frac{1}{\sqrt{2}}, \cos(nt) >= 0, \quad \forall n \neq 0$

(vi) $< \frac{1}{\sqrt{2}}, \sin(nt) >= 0, \quad \forall n$

$\Longrightarrow$ We can see that:

$\{\cos(nt)\}_{n=1}^{\infty}$, $\{\sin(nt)\}_{n=1}^{\infty}$, and $\dfrac{1}{\sqrt{2}}$ constitute an *orthonormal* basis for **F**

FACT:
In general, the magnitude(or contribution) of each element $\{f_n(t)\}_{n=1}^{N}$ of an *orthonormal* basis for a signal $x(t)$ in a signal space **F** is the *projection* of $x(t)$ onto $f_n(t)$'s, and the projection is done by taking the inner product between $x(t)$ and $f_n(t)$.

Example 3.4

A vector $\vec{a} = (a_1, a_2, a_3)$ in $\mathbf{R}^3$ space.

Example 3.5

A signal $x(t) = 1 + 2\sin(t) + 3\cos(t)$ in the signal space $\mathbf{F}$.

Now, for any signal $x(t)$ in $\mathbf{F}$,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \in \mathbf{F}$$

(1) Projection of $x(t)$ onto $\frac{1}{\sqrt{2}}$: provide a_0 (d.c. component)

$$< x(t), \frac{1}{\sqrt{2}} > \quad = \quad < \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt), \frac{1}{\sqrt{2}} >$$

$$\text{LHS} =< x(t), \frac{1}{\sqrt{2}} >= \frac{1}{\sqrt{2}} \frac{1}{\pi} \int_0^{2\pi} x(t)dt$$

$$\text{RHS} =< \frac{a_0}{2}, \frac{1}{\sqrt{2}} >= \frac{1}{\pi} \int_0^{2\pi} \frac{a_0}{2\sqrt{2}} dt = \frac{a_0}{\sqrt{2}}$$

$$\Longrightarrow \quad \frac{a_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{1}{\pi} \int_0^{2\pi} x(t)dt$$

$$\Longrightarrow \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} x(t)dt$$

(2) Projection of $x(t)$ onto $\{\cos(mt)\}_{m=1}^{\infty}$: provide a_m's

$$\begin{aligned} < x(t), \cos(mt) > \quad &= \quad < \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt), \cos(mt) > \\ &= \quad < \frac{a_0}{2} + \cos(mt) > + \sum_{n=1}^{\infty} < a_n \cos(nt), \cos(mt) > \\ &\qquad + \sum_{n=1}^{\infty} < b_n \sin(nt), \cos(mt) > \end{aligned}$$

$$\text{LHS} =< x(t), \cos(mt) >= \frac{1}{\pi} \int_0^{2\pi} x(t) \cos(mt)dt$$

$$\text{RHS} =< \cos(mt), \cos(mt) >= \frac{1}{\pi} \int_0^{2\pi} \cos^2(mt)dt = a_m$$

$$\Longrightarrow \quad a_m = \frac{1}{\pi} \int_0^{2\pi} x(t) \cos(mt)dt, \quad m = 1, 2, 3, \cdots$$

Note:
Notice that the above formula of a_m can be used including the d.c. componenet, i.e., for $m = \mathbf{0}, 1, 2, 3, \cdots$.

(3) Projection of $x(t)$ onto $\{\sin(mt)\}_{m=1}^{\infty}$: provide b_m's

Similarly, we get b_m's as follows:

$$b_m = \frac{1}{\pi} \int_0^{2\pi} x(t) \sin(mt) dt, \quad m = 1, 2, 3, \cdots$$

derivation: assignment

NOTE:
In either way, MSE minimization or orthonormal basis for signal space, we get the same formula for the F.S. coefficients!!!

Special properties of F.S. coefficients

(1) Symmetric signal (even function): $x(t) = x(-t)$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(t)\cos(nt)dt$$
$$b_n = 0 \qquad \forall n = 1, 2, \cdots$$

Figure 3.2: Symmetric signal $x(t)$

(cf.) Symmetric signals can be expessed using only cosine terms, including the d.c. component

(2) Asymmetric signal (odd function): $x(t) = -x(-t)$

$$a_n = 0 \qquad \forall n = 1, 2, \cdots$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} x(t)\sin(nt)dt$$

Figure 3.3: Asymmetric signal $x(t)$

(cf.) Asymmetric signals can be expessed using only sine terms, and the d.c. component does not exists inherently.

Proof:

$$
\begin{aligned}
\textbf{A}.\ a_n &= \frac{1}{\pi}\int_{2\pi} x(t)\cos(nt)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\cos(nt)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x(t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt \ \ (\text{let } t^{'} = -t \text{ in 1st term}) \\
&= \frac{1}{\pi}\int_{\pi}^{0} x(-t^{'})\cos(-nt^{'})(-dt^{'}) + \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} x(-t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt
\end{aligned}
$$

(1) Symmetric $x(t)$: $x(t) = x(-t)$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt$$

(2) Asymmetric $x(t)$: $x(t) = -x(-t)$

$$a_n = -\frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\cos(nt)dt = 0$$

$$
\begin{aligned}
\textbf{B}.\ b_n &= \frac{1}{\pi}\int_{2\pi} x(t)\sin(nt)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x(t)\sin(nt)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x(t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt \ \ (\text{let } t^{'} = -t \text{ in 1st term}) \\
&= \frac{1}{\pi}\int_{\pi}^{0} x(-t^{'})\sin(-nt^{'})(-dt^{'}) + \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt \\
&= -\frac{1}{\pi}\int_{0}^{\pi} x(-t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt
\end{aligned}
$$

(1) Symmetric $x(t)$: $x(t) = x(-t)$

$$b_n = -\frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt = 0$$

(2) Asymmetric $x(t)$: $x(t) = -x(-t)$

$$b_n = \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt = \frac{2}{\pi}\int_{0}^{\pi} x(t)\sin(nt)dt$$

F.S. representation of a symmetric signal $x(t)$

Intuition:
Symmetric signals can be expressed in a F.S. form using only symmetric(i.e. d.c. and cosine) terms, whereas asymmetric signals need only asymmetric(i.e. sine) terms!!!

Gibb's Phenomenon

Now, we have a Fourier series representation of a periodic signal $x(t)$ as follows:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \tag{3.7}$$

Notice that:

♣ Practically, we cannot use infinite number of coefficients $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ for (3.7).

♣ Therefore, we have to use a finite number of a_n's and b_n's, i.e.

$$\hat{x}(t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt) \tag{3.8}$$

This is a *truncated* Fourier series, and due to the difference(or error) between (3.7) and (3.8), there inevitably happens some overshoots and undershoots in $\hat{x}(t)$.

$\Longrightarrow$ This is called the **"GIBB's PHENOMENON"**

Example 3.6

Figure 3.4: A cosine square wave signal.

NOTE:
The more coefficients we use, the better (or closer) Fourier series representation of $x(t)$ by $\hat{x}(t)$!!!

Figure 3.5: Effect of the number of F.S. coefficient.

3.4 Complex Representation of Fourier Series

Under the same assumptions(i.e. periodicity and Dirichlet conditions) on $x(t)$ as in the trigonometric representation of Fourier series for $x(t)$, we can express $x(t)$ with a linear combination of harmonically related *complex exponentials*, i.e.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{jkt} \quad \text{where } T_0 = 2\pi(\text{sec})$$

and the corresponding complex F.S. coefficients $\{C_k\}_{k=-\infty}^{\infty}$ are given:

$$C_k = \frac{1}{2\pi} \int_{2\pi} x(t) e^{-jkt} dt$$

Note:

In general, C_k's are complex numbers, i.e.:

$$\begin{aligned} C_k &= \text{Re}[C_k] + j\text{Im}[C_k] \ \ \text{(cartesian coordinate)} \\ &= |C_k| e^{j\Phi_k} \ \ \text{(polar coordinate)} \quad : \textit{preferred}!!! \end{aligned}$$

where

$$|C_k| = \sqrt{\text{Re}^2[C_k] + \text{Im}^2[C_k]}$$

$$\Phi_k = \arctan\left\{\frac{\text{Im}[C_k]}{\text{Re}[C_k]}\right\}$$

Figure 3.6: Polar form of complex F.S. coefficient C_k

Derivation of complex Fourier series

From the well known Euler's formula:

$$e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$$

We have

$$e^{\pm jnt} = \cos(nt) \pm j\sin(nt)$$

$$\Longrightarrow \begin{cases} \cos(nt) = \frac{1}{2}\left(e^{jnt} + e^{-jnt}\right) \\ \sin(nt) = \frac{1}{2j}\left(e^{jnt} - e^{-jnt}\right) \end{cases}$$

Then, the triginometric F.S. representation of $x(t)$ becomes:

$$\begin{aligned} x(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos(nt) + b_n\sin(nt)\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n\left(\frac{e^{jnt}+e^{-jnt}}{2}\right) + b_n\left(\frac{e^{jnt}-e^{-jnt}}{2j}\right)\right\} \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{\frac{1}{2}(a_n - jb_n)e^{jnt} + \frac{1}{2}(a_n + jb_n)e^{-jnt}\right\} \quad (\textbf{cf}.\ \frac{1}{j} = -j) \\ &\quad (\text{ let } n = k \text{ and } n = -k \text{ in 1st and 2nd sum respectively}) \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty}\frac{1}{2}(a_k - jb_k)e^{jkt} + \sum_{k=-1}^{-\infty}\frac{1}{2}(a_{-k} + jb_{-k})e^{jkt} \end{aligned} \tag{3.9}$$

Fact: Now, let's take a look at the properties of a_k and b_k for a moment, and we can check the following facts:

$$\begin{cases} a_{-k} = a_k \\ b_{-k} = -b_k \\ b_0 = 0 \end{cases} \tag{3.10}$$

proof:

$$a_{-k} = \frac{1}{\pi}\int_{2\pi} x(t)\cos(-kt)dt = \frac{1}{\pi}\int_{2\pi} x(t)\cos(kt)dt \triangleq a_k$$

$$b_{-k} = \frac{1}{\pi}\int_{2\pi} x(t)\sin(-kt)dt = -\frac{1}{\pi}\int_{2\pi} x(t)\sin(kt)dt \triangleq -b_k$$

$$b_0 = \frac{1}{\pi}\int_{2\pi} x(t)\sin(0)dt = 0 \qquad \textbf{q.e.d}.$$

Using (3.10), (3.9) can be expressed as:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{1}{2}(a_k - jb_k)e^{jkt} + \sum_{k=-1}^{-\infty} \frac{1}{2}(a_k - jb_k)e^{jkt} \tag{3.11}$$

Now, let

$$C_k \triangleq \frac{1}{2}(a_k - jb_k)$$

Then,

$$\begin{cases} C_0 = \frac{1}{2}(a_0 - jb_0) = \frac{1}{2}a_0 \quad (\textbf{cf.}\ b_0 = 0) \\ e^{j0t} = e^0 = 1 \end{cases}$$

Therefore, (3.11) becomes:

$$\begin{aligned} x(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} C_k e^{jkt} + \sum_{k=-1}^{-\infty} C_k e^{jkt} \\ &= \sum_{k=-\infty}^{\infty} C_k e^{jkt} \end{aligned}$$

: Complex F.S. representation

Corresponding coefficient C_k is given by:

$$\begin{aligned} C_k &\triangleq \frac{1}{2}(a_k - jb_k) \\ &= \frac{1}{2}\left\{\frac{1}{\pi}\int_0^{2\pi} x(t)\cos(kt)dt - j\frac{1}{\pi}\int_0^{2\pi} x(t)\sin(kt)dt\right\} \\ &= \frac{1}{2\pi}\int_0^{2\pi} x(t)\left\{\cos(kt) - j\sin(kt)\right\}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} x(t)e^{-jkt}dt \end{aligned}$$

NOTE:
In general, with an arbitrary period T, i.e.

$$x(t) = x(t+T)$$

then, the complex Fourier series representation of $x(t)$ is given as:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{j\frac{2\pi kt}{T}}$$

$$C_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}\,dt$$

which is called the **"General Expression for a Complex F.S."**

Remarks:

1. Once we compute C_k's for positive k's ($k = 1, 2, \cdots$), then C_{-k}'s can be readily obtained as:

$$C_{-k} = C_k^*$$

proof:

$$\begin{aligned} C_{-k} &= \frac{1}{2}(a_{-k} - jb_{-k}) \\ &= \frac{1}{2}(a_k + jb_k) \quad (a_k \text{ is even, and } b_k \text{ is odd w.r.t. k}) \\ &= \left\{\frac{1}{2}(a_k - jb_k)\right\}^* \\ &= C_k^* \end{aligned}$$

2. If $x(t)$ is a symmetric signal of t (i.e. $x(t) = x(-t)$), then C_k is *pure real*, i.e. $C_k = \text{Re}[C_k]$, since b_k's are all zero for even function of t.

3. If $x(t)$ is an asymmetric signal of t (i.e. $x(t) = -x(-t)$), then C_k is *pure imaginary*, i.e. $C_k = j\text{Im}[C_k]$, since a_k's are all zero for odd function of t.

4. Let's define the general expression for the complex exponentials as:

$$\phi_k(t) \triangleq \frac{1}{\sqrt{2\pi}} e^{jkt}$$

Then, if we define the inner product between $\phi_n(t)$ and $\phi_m(t)$ as:

$$< \phi_n(t), \phi_m(t) > \triangleq \int_0^{2\pi} \phi_n(t)\phi_m^*(t)dt$$

We can check $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ form an orthonormal basis for a signal space $\mathbf{F}_1$, which is composed of periodic signals represented by complex F.S.:

$$\mathbf{F}_1 = \left\{ x(t) | x(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{jkt} \right\}$$

Assignment:

(a) Show that $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ form a basis for $\mathbf{F}_1$.

(b) Prove the orthonormality between $\phi_n(t)$ and $\phi_m(t)$.

(c) Derive C_k using projection onto each basis signal(i.e. taking inner product between $x(t)$ and each $\{\phi_k(t)\}_{k=-\infty}^{\infty}$

derivation of C_k:

Example 3.7

Determine whether the signal $x(t)$ given below can be expressed in a Fourier series, and if it does have its own F.S., find the complex Fourier series coefficient C_k of it.

$$x(t) = \begin{cases} 1, & -1 \le t < 1 \\ 0, & 1 \le t < 3 \end{cases} \quad \text{and } T = 4 \text{ (sec)}$$

Figure 3.7: A train of pulses $x(t)$

Remarks: Notice that $x(t)$ is an even function of t, and thus

(i) $b_k = 0$, i.e. C_k must be *pure real*!

(ii) d.c. component of $x(t)$ is obviously $\frac{1}{2} = C_0$ from above!

Solution:

discussion:

(a) Notice that C_k indeed is pure real, and $C_0 = \frac{1}{2}$. [(i), and (ii)]

(b) Check that $C_{-k} = C_k^* = C_k$, i.e.

$$C_{-k} = \frac{1}{2}\text{sinc}(\frac{-k}{2}) = \frac{1}{2}\text{sinc}(\frac{k}{2}) = C_k = C_k^*$$

Figure 3.8: Complex F.S. coefficient C_k of cosine square wave $x(t)$

Example 3.8

Repeat the above example for the following $x(t)$.

$$x(t) = \begin{cases} 1, & 0 \le t < \pi \\ -1, & -\pi \le t < 0 \end{cases} \quad \text{and } T = 2\pi \text{ (sec)}$$

Figure 3.9: A sine square wave $x(t)$

Remarks: Notice that $x(t)$ is now an odd function of t, and thus

(i) $a_k = 0$, i.e. C_k must be *pure imaginary*!

(ii) d.c. component of $x(t)$ is obviously $C_0 = 0$ from above!

Solution:

discussion:

(a) Notice that C_k indeed is pure imaginary, and $C_0 = 0$ [(i), and (ii)], i.e.:

$$C_0 = \lim_{k \to 0} \frac{1 - \cos(\pi k)}{j\pi k} = \lim_{k \to 0} \frac{\pi \sin(\pi k)}{j\pi} = 0 \text{ (by L'Hospital's law)}$$

(b) Check that $C_{-k} = C_k^*$, i.e.

$$C_{-k} = j \cdot \frac{\cos(-\pi k) - 1}{-\pi k} = -j \cdot \frac{\cos(\pi k) - 1}{\pi k} = C_k^*$$

Figure 3.10: Complex F.S. coefficient C_k of sine square wave $x(t)$