Contents

Chapter 4

FOURIER TRANSFORM

4.1 Concept of Fourier Transform

Suppose we have an *aperiodic* signal $x(t)$ which satisfies the following conditions:

Figure 4.1: Non-periodic signal $x(t)$

Dirichlet conditions:

1. $x(t)$ is *absolutely integrable*, i.e.

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

2. $x(t)$ has a *finite* number of *finite* maxima and minima $\forall\, t$.

3. $x(t)$ has a *finite* number of *finite* discontinuities $\forall\, t$.

Then, the Fourier transform pair of $x(t)$ is given by:

$$X(\omega) = \mathcal{F}\left[x(t)\right] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \mathcal{F}^{-1}\left[X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$\left(x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad : \textbf{Fourier transform pair}\right)$$

Note:

(1) $X(\omega)$ represents the frequency distribution of a non-periodic signal $x(t)$, as the Fourier series coefficient C_k represents that of periodic signals.

(2) The difference, however, is that non-periodic $x(t)$ may have any frequency component (i.e., ω is *continuous*), whereas periodic $x(t)$ can only have *harmonic* frequencies(i.e., $k\omega_0$ where $\omega_0 = \frac{2\pi}{T}$(rad/sec))

Derivation of F.T pair

Let's define a periodic signal $\tilde{x}(t)$, which is the repetition of non-periodic $x(t)$ with an arbitrary period T, i.e.:

Figure 4.2: Periodic signal $\tilde{x}(t)$ from $x(t)$

Then, $\tilde{x}(t)$ can be expressed as a Fourier series, since all the Dirichlet condtions are met for $\tilde{x}(t)$ within its period, $-\frac{T}{2} \le t < \frac{T}{2}$, and we have the following fact:

$$\lim_{T \to \infty} \tilde{x}(t) = x(t)$$

Basic idea:
Compute C_k for $\tilde{x}(t)$, and let $T \to \infty$ This will provide us with some idea on the analysis of frequency components in $x(t)$!!!

(1) The Fourier series coefficient for $\tilde{x}(t)$ is given by:

$$
\begin{aligned}
C_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t)e^{-j\frac{2\pi k}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}dt \qquad (4.1)
\end{aligned}
$$

Note:
It is meaningless just to let $T \to \infty$ for C_k since $\lim_{T\to\infty} C_k = 0$, and this is due to the fact that $x(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt < \int_{-\infty}^{\infty} |x(t)e^{-j\omega t}|dt < \infty$$

Let's define $X(k) \triangleq T \cdot C_k$, and let $\Delta\omega = \frac{2\pi}{T}$, i.e.

$$
\begin{aligned}
X(k) \triangleq T \cdot C_k &= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}dt \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\Delta\omega kt}dt \qquad (4.2)
\end{aligned}
$$

where $\Delta\omega = \frac{2\pi}{T}$(rad/sec) is the fundamental frequency of $\tilde{x}(t)$.

Now, as $T \to \infty$, $\Delta\omega \to 0$ and we have:

$$\lim_{T\to\infty} \Delta\omega \cdot k \left(= \lim_{T\to\infty} \frac{2\pi k}{T} \right) \stackrel{d}{=} \omega$$

Meaning:
The *discrete* variable k becomes a *continuous* variable ω, which means that the harmonic frequencies $\Delta\omega \cdot k$ become *continuous*!!!

Therefore, (4.2) becomes:

$$
\begin{aligned}
X(\omega) \triangleq \lim_{T\to\infty} X(k) &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\Delta\omega kt}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad (4.3)
\end{aligned}
$$

We call (4.3), the **Fourier Transform** $X(\omega)$ of $x(t)$.

(2) Now, in order to compute $x(t)$ back from $X(\omega)$, we use the following fact, discussed earlier in this section:

$$x(t) = \lim_{T\to\infty} \tilde{x}(t)$$

Here,

$$\begin{aligned}
\tilde{x}(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi kt}{T}} &= \sum_{k=-\infty}^{\infty} \frac{X(k)}{T} e^{j\Delta\omega kt} \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k) e^{j\Delta\omega kt} \cdot \Delta\omega \\
& \quad (\Delta\omega = \frac{2\pi}{T} \to \frac{1}{T} = \frac{\Delta\omega}{2\pi})
\end{aligned}$$

On the other hand, as $T \to \infty$, the following facts hold:

1. $\lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$
2. $X(k) \longrightarrow X(\omega)$
3. $\Delta\omega k \longrightarrow \omega$
4. $\Delta\omega \longrightarrow d\omega$

Therefore,

$$\begin{aligned}
x(t) = \lim_{T\to\infty} \tilde{x}(t) &= \frac{1}{2\pi} \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} X(k) e^{j\Delta\omega kt} \cdot \Delta\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \qquad (4.4)
\end{aligned}$$

We call (4.4), the **Inverse Fourier Transform** $x(t)$ of $X(\omega)$.

In summary, we have the following Fourier transform pair of a non-periodic signal $x(t)$:

$$\mathcal{F}[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\mathcal{F}^{-1}[X(\omega)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Remarks:

1. Usually, the Fourier transform $X(\omega)$ is a complex quantity, i.e.

$$X(\omega) = \mathrm{Re}[X(\omega)] + j\mathrm{Im}[X(\omega)] = |X(\omega)|e^{j\Phi_X(\omega)}$$

   and $|X(\omega)|$ and $\Phi_X(\omega)$ are called the *magnitude* spectrum and the *phase* spectrum of $X(\omega)$ respectively.

2. The reason why we need transforms of signal such as F.S. and F.T. is that these transforms allow us an easy manipulation on given signals and systems to achieve specific goals.(it will be discussed later with some examples of application)

3. The dimension of the frequency ω is rad/sec, since $\omega = \lim_{T\to\infty} \Delta\omega \cdot k$ where $\Delta\omega$ is in (rad/sec) and k is *dimensionless*.

Another form of F.T.:

If you prefer the *cyclic* frequency f(Hz) rather than the *angular* frequency ω(rad/sec), the F.T. pair is in the following forms:

$$\mathcal{F}[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\mathcal{F}^{-1}[X(f)] = x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

derivation:

By way of change of variable as follows:

$$\omega = 2\pi f$$

then, $d\omega = 2\pi df$ and

$$X(\omega) \Longrightarrow X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\begin{aligned} x(t) \Longrightarrow x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}2\pi df \\ &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \end{aligned}$$

4.2 Characteristics of Fourier Transform

(1) Linearity

$$\begin{aligned}\mathcal{F}[ax(t)+by(t)] &= a\mathcal{F}[x(t)]+b\mathcal{F}[y(t)] \\ &= aX(\omega)+bY(\omega)\end{aligned}$$

proof:

$$\begin{aligned}\mathrm{LHS}=\mathcal{F}[ax(t)+by(t)] &= \int_{-\infty}^{\infty}\{ax(t)+by(t)\}e^{-j\omega t}dt \\ &= a\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt+b\int_{-\infty}^{\infty}y(t)e^{-j\omega t}dt \\ &= aX(\omega)+bY(\omega) \\ &= \mathrm{RHS}\end{aligned}$$

(2) Shift in time domain

$$\mathcal{F}[x(t-a)] = X(\omega)e^{-ja\omega}$$

proof:

$$\begin{aligned}\mathrm{LHS}=\mathcal{F}[x(t-a)] &= \int_{-\infty}^{\infty}x(t-a)e^{-j\omega t}dt \\ &\quad (\text{let } \ t^{'}=t-a) \\ &= \int_{-\infty}^{\infty}x(t^{'})e^{-j\omega(t^{'}+a)}dt^{'} \\ &= \int_{-\infty}^{\infty}x(t^{'})e^{-j\omega t^{'}}\cdot e^{-j\omega a}dt^{'} \\ &= e^{-j\omega a}\cdot\int_{-\infty}^{\infty}x(t^{'})e^{-j\omega t^{'}}dt^{'} \\ &= X(\omega)e^{-ja\omega} \\ &= \mathrm{RHS}\end{aligned}$$

note:
Since $\mathcal{F}[x(t-a)] = X(\omega)e^{-ja\omega} = |X(\omega)|e^{j[\Phi_X(\omega)-a\omega]}$, there happens only the *phase delay*, i.e. $\Delta\phi = -a\omega$, and the magnitude spectrum does not change!

(3) Scaling in time domain

$$\mathcal{F}[x(at)] = \frac{1}{a}X(\frac{\omega}{a}) \quad \text{where } a > 0$$

proof:

$$\begin{aligned}
\text{LHS} = \mathcal{F}[x(at)] &= \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt \\
&\quad (\text{let } \ t^{'} = at, \ \ \text{then } \ dt^{'} = adt) \\
&= \int_{-\infty}^{\infty} x(t^{'})e^{-j\omega\frac{t^{'}}{a}} \cdot \frac{1}{a}dt^{'} \\
&= \frac{1}{a}\int_{-\infty}^{\infty} x(t^{'})e^{-j(\frac{\omega}{a})t^{'}}dt^{'} \\
&\triangleq \frac{1}{a}X(\frac{\omega}{a}) \\
&= \text{RHS}
\end{aligned}$$

note:
Notice that the time domain compression($a > 1$) corresponds to the frequency domain stretching, and the time domain stretching($0 < a < 1$) corresponds to the frequency domain compression, i.e.:

$$\textbf{Time compression (stretching)} \longleftrightarrow \textbf{Frequency stretching (compression)}$$

e.g.:

Figure 4.3: Scaling in time domain.

remark:
Intuitively, $x_1(t)(x_2(t))$ represents the increase(decrease) of frequency both in time and frequency domains, compared to $x_0(t)$!!!

(4) Convolution

$$\mathcal{F}[x(t) * y(t)] = X(\omega) \cdot Y(\omega)$$

proof:

$$\begin{aligned}
\text{LHS} = \mathcal{F}[x(t) * y(t)] &= \int_{-\infty}^{\infty} x(t) * y(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \right\} e^{-j\omega t} dt \\
&\quad \text{(interchange the order of integration)} \\
&= \int_{-\infty}^{\infty} x(\tau) \left\{ \int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t} dt \right\} d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) Y(\omega) e^{-j\omega\tau} d\tau \\
&\quad \text{(by the property (2): shift in time)} \\
&= \left[\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \right] Y(\omega) \\
&= X(\omega) \cdot Y(\omega) \\
&= \text{RHS}
\end{aligned}$$

note:
For an LTI system,

Figure 4.4: An LTI system.

Let

$$\begin{aligned}
X(\omega) &= \mathcal{F}[x(t)] \\
Y(\omega) &= \mathcal{F}[y(t)] \\
H(\omega) &= \mathcal{F}[h(t)]
\end{aligned}$$

where $H(\omega) = \mathcal{F}[h(t)]$ is called the *transfer function* of the system, then notice that the input/output relationships are in the following forms:

$$\begin{cases} y(t) = h(t) * x(t) & \text{time domain} \\ \\ Y(\omega) = H(\omega) X(\omega) & \text{frequency domain} \end{cases}$$

$$\Longrightarrow \quad y(t) = \mathcal{F}^{-1}\left\{ H(\omega) X(\omega) \right\}$$

(5) Duality

$$\text{Let } \mathcal{F}[x(t)] = X(\omega), \quad \text{then}$$
$$\mathcal{F}[X(t)] = 2\pi x(-\omega)$$
$$\text{or}$$
$$\mathcal{F}[X(-t)] = 2\pi x(\omega)$$

proof:

We are given the following F.T. pair:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{4.5}$$
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{4.6}$$

Change the role of t and ω in (4.6), then,

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{j\omega t} dt$$
$$\Downarrow$$
$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt \equiv \frac{1}{2\pi} \mathcal{F}[X(t)]$$

Therefore,

$$\mathcal{F}[X(t)] = 2\pi x(-\omega)$$

(cf.) If we carry out the same procedure for (4.5), we get:

$$X(t) = \int_{-\infty}^{\infty} x(\omega) e^{-j\omega t} d\omega$$
$$\Downarrow$$
$$X(-t) = \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega \equiv 2\pi \mathcal{F}^{-1}[x(\omega)]$$

Therefore,

$$\mathcal{F}[X(-t)] = 2\pi x(\omega)$$

note: Once we know $X(\omega) = \mathcal{F}[x(t)]$, then the Fourier transform of $X(t)$ can easily be calculated from the duality property!!!

(6) Singular functions(Generalized Fourier transform)

Some special functions, called *singular functions*, have their own Fourier transforms even though they do not obey the Dirichlet conditions:

<u>Dirichlet conditions;</u>

1. $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
2. (i) $x(t)$ has finite number of finite maxima and minima
   (ii) $x(t)$ has finite number of finite discontinuities

(6-1) Unit impulse function: $\delta(t)$

violates the 2-(i) and 2-(ii) conditions, but

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Figure 4.5: Unit impulse function

$$\begin{aligned}
\mathcal{F}[\delta(t)] &= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \delta(t)e^{-j\omega 0}dt \\
&= \int_{-\infty}^{\infty} \delta(t)dt \\
&= 1
\end{aligned}$$

Therefore, even though $\delta(t)$ violates the Dirichlet condition, it has its own Fourier transform as follows:

$$\mathcal{F}[\delta(t)] = 1 \quad \text{or} \quad \mathcal{F}^{-1}[1] = \delta(t)$$

remarks: The existence of $\mathcal{F}[\delta(t)]$ comes from the facts below:

1. The sifting property of $\delta(t)$, i.e.

$$\int_{-\infty}^{\infty} \delta(t-\alpha)g(t)dt = g(\alpha)$$

$$\begin{aligned} LHS &= \int_{-\infty}^{\infty} \delta(t-\alpha)g(t)dt \\ &= g(\alpha)\int_{-\infty}^{\infty} \delta(t-\alpha)dt = g(\alpha) = RHS \end{aligned}$$

Figure 4.6: Sifting property of unit impulse function

2. The area of $\delta(t)$ is one,i.e.

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

Question: What is the Fourier transform of a d.c. signal,i.e., $\mathcal{F}[1]$?

Answer:

Physical meaning, interpretation, and example:

(i) $\mathcal{F}[\delta(t)] = 1$

white noise: contains every frequency component with equal magnitude

Figure 4.7: Unit impulse function and its F.T.

Example 4.1

Radio receiver when lightning strikes:

Figure 4.8: Radio broadcasting and receiver

(ii) $\mathcal{F}[1] = 2\pi\delta(\omega)$

d.c. signal contains only the zero frequency component
(cf.) magnitude of $X(\omega)$= area of $X(\omega) = 2\pi$

Figure 4.9: A d.c. signal and its F.T.

(cf.) In a similar way, the Fourier series of $x(t)$, which is the repetition of $\delta(t)$ with period 2π cannot be computed in principle since it violates the Dirichlet conditions. However, if we take the Fourier series of $x(t)$, we get

Figure 4.10: The unit impulse train signal

$$\begin{aligned} C_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t)e^{-jkt}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(t)dt \\ &= \frac{1}{2\pi} \quad \forall\, k \end{aligned}$$

Therefore, the impulse train signal can be put into a Fourier series as follows:

$$\begin{aligned} x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jkt} &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} e^{jkt} \\ &= \frac{1}{2\pi} + 2\sum_{k=1}^{\infty} \frac{1}{2\pi}\{\cos(kt) + j\sin(kt)\} \\ &= \frac{1}{2\pi} + \sum_{k=1}^{\infty} \frac{\cos(kt)}{\pi} \end{aligned}$$

note: In above derivation, following facts are applied:
(a) Since C_k is real, $C_{-k} = C_k^* = C_k$, ($x(t)$ is a symmetric signal!)
(b) $\sin(-kt) = -\sin(kt)$ terms cancel out $\sin(kt)$ terms.

Figure 4.11: The F.S. representation of unit impulse train signal

(6-2) Unit step function: $u(t)$

violates the condition 1(absolute integrability)

Figure 4.12: Unit step function

If you try to compute $\mathcal{F}[u(t)]$ directly, you get

$$\mathcal{F}[u(t)] = \int_0^{\infty} 1 \cdot e^{-j\omega t} dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_0^{\infty} = \frac{1}{j\omega} - \frac{e^{-j\omega\infty}}{j\omega} \quad : \text{non-sense}$$

To compute $\mathcal{F}[u(t)]$, we first define the signum function $\text{sgn}(t)$, and compute its Fourier transform:

Define,

$$\text{sgn}(t) \triangleq \begin{cases} 1, & t \geq 0 \\ -1 & t < 0 \end{cases}$$

note that d.c. $= 0$

Figure 4.13: Signum function $\text{sgn}(t)$

note:

1. Notice that $u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$, and if there $\exists\ \mathcal{F}[\text{sgn}(t)]$, then

$$U(\omega) = \mathcal{F}[\frac{1}{2}] + \frac{1}{2}\mathcal{F}[\text{sgn}(t)] \quad : \text{by linearity}$$

2. But, $\text{sgn}(t)$ still violates the *absolute integrability* condition

Let's now consider the following $x(t)$, where $a > 0$.

$$x(t) \triangleq \begin{cases} e^{-at}, & t \geq 0 \\ -e^{at} & t < 0 \end{cases}$$

Then, $x(t)$ is absolutely integrable [1], and its Fourier transform thus can be calculated.

note that d.c. $= 0$, and

$$\lim_{a \to 0} x(t) = \text{sgn}(t)$$

Figure 4.14: approximation of sgn(t)

After we get $\mathcal{F}[x(t)]$, we send $a \to 0$, then

$$\lim_{a \to 0} \mathcal{F}[x(t)] = \mathcal{F}[\text{sgn}(t)]$$

Now,

$$\begin{aligned}
\mathcal{F}[x(t)] &= \int_{-\infty}^{0} e^{at}(-1)e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}(1)e^{-j\omega t}dt \\
&= -\int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt \\
&= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty} - \left.\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right|_{-\infty}^{0} \\
&= \frac{1}{a+j\omega} - \frac{1}{a-j\omega} \\
&= \frac{-2\omega j}{a^2+\omega^2}
\end{aligned}$$

Therefore,

$$\mathcal{F}[\text{sgn}(t)] = \lim_{a \to 0} \mathcal{F}[x(t)] = \lim_{a \to 0}\left(\frac{-2\omega j}{a^2+\omega^2}\right) = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

[1] $\int_{-\infty}^{\infty} |x(t)|dt = 2\int_{0}^{\infty} e^{-at}dt = 2\left.\frac{e^{-at}}{-a}\right|_{0}^{\infty} = \frac{2}{a}$.

(cf.) Notice that $X(\omega)$ and $\mathcal{F}[\text{sgn}(t)]$ are *pure imaginary*, and it is because the signals are odd functions of t: this property is not discussed officially in class, but you can easiliy prove it following the similar procedure discussed at the special property of Fourier series coefficient!!!

Figure 4.15: Fourier transform of sgn(t) along with $X(\omega)$

Now that

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

$$\begin{aligned}
\mathcal{F}[u(t)] &= \mathcal{F}[\frac{1}{2}] + \frac{1}{2}\mathcal{F}[\text{sgn}(t)] \\
&= \frac{1}{2}2\pi\delta(\omega) + \frac{1}{2}\frac{2}{j\omega} \\
&= \pi\delta(\omega) + \frac{1}{j\omega}
\end{aligned}$$

i.e.,

$$\mathcal{F}[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Figure 4.16: Fourier transform of unit step function $u(t)$

Interpretation:

Since $\mathcal{F}[x(t)] = X(\omega) = 0$ at $\omega = 0$,

$$\mathcal{F}[\mathrm{sgn}(t)] = \lim_{a \to 0} X(\omega) = \begin{cases} 0, & \omega = 0 \\ \frac{2}{j\omega} & \omega \neq 0 \end{cases}$$

Therefore,

$$\begin{aligned} \mathcal{F}[u(t)] &= \mathcal{F}[\frac{1}{2}] + \frac{1}{2}\mathcal{F}[\mathrm{sgn}(t)] \\ &= \begin{cases} \mathcal{F}[\frac{1}{2}] = \pi\delta(\omega), & \omega = 0 \\ \frac{1}{2}\mathcal{F}[\mathrm{sgn}(t)] = \frac{1}{j\omega} & \omega \neq 0 \end{cases} \end{aligned}$$

Note: Relationship among $\delta(t)$, $u(t)$, and $r(t)$, where the *ramp function* $r(t)$ is defined as follows: [2]

$$r(t) \triangleq \begin{cases} t, & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Figure 4.17: The ramp function $r(t)$

(cf.) For later use, check out and keep in mind the following relations:

$$(1)\;\; u(t) = \int_{-\infty}^{t} \delta(\theta) d\theta = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}, \qquad \frac{du(t)}{dt} = \delta(t)$$

$$(2)\;\; r(t) = \int_{-\infty}^{t} u(\theta) d\theta = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}, \qquad \frac{dr(t)}{dt} = u(t)$$

[2] Since the ramp function violates the absolute integrability, it is also a singular function.

(7) Differentiation

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega)$$

proof:

We have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

then

$$\begin{aligned}
\frac{dx(t)}{dt} &= \frac{d}{dt}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left\{\frac{d}{dt}e^{j\omega t}\right\}d\omega \\
&\quad \text{(by Leibniz's rule: (cf.))} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{X(\omega)\cdot j\omega\}\, e^{j\omega t}d\omega \\
&\triangleq \mathcal{F}^{-1}\left[j\omega X(\omega)\right]
\end{aligned}$$

Therefore,

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega)$$

(cf.) Leibniz's rule

Let

$$g(x) \triangleq \int_{\alpha(x)}^{\beta(x)} f(x,u)du$$

where $f(x,u)$ is a continuous function with respect to x and u.
Then,

$$\frac{dg(x)}{dx} = f(x,\beta(x))\frac{d\beta(x)}{dx} - f(x,\alpha(x))\frac{d\alpha(x)}{dx} + \int_{\alpha(x)}^{\beta(x)}\frac{\partial}{\partial x}f(x,u)du$$

Example 4.2

Cross-check the Fourier transform of $\delta(t)$ using the differentiation property.

Solution:

NOTE: Another simplified way of computing $\mathcal{F}[u(t)]$

Figure 4.18: Unit step function

Define

$$x(t) \triangleq -\frac{1}{2} + u(t) \tag{4.7}$$

Figure 4.19: Scaled signum function

Notice that

$$\frac{dx(t)}{dt} = \delta(t)$$

Therefore,

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega) \equiv \mathcal{F}[\delta(t)] = 1$$

$$\Longrightarrow X(\omega) = \frac{1}{j\omega} \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{aligned} X(\omega) = \mathcal{F}[-\frac{1}{2} + u(t)] &= -\frac{1}{2}2\pi\delta(\omega) + U(\omega) \\ &\equiv \frac{1}{j\omega} \end{aligned}$$

Therefore,

$$U(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

(8) Integration

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\theta)d\theta\right] = \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)X(\omega) = U(\omega)X(\omega)$$

proof:

Let

$$y(t) = \int_{-\infty}^{t} x(\theta)d\theta$$

then, $y(t)$ is the output signal of an LTI system, whose impulse response is the unit step function, i.e., $h(t) = u(t)$:

$$y(t) = h(t) * x(t) = u(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

Figure 4.20: Output of an LTI, where $h(t) = u(t)$

Therefore,

$$\begin{aligned} Y(\omega) &= \mathcal{F}[u(t)] \cdot \mathcal{F}[x(t)] \\ &= \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)X(\omega) \end{aligned}$$

Example 4.3

Find the Fourier transform of the ramp function $r(t)$.

Solution:

check:

$$\frac{dr(t)}{dt} = u(t)$$

$$\longrightarrow\ \mathcal{F}\left[\frac{dr(t)}{dt}\right] = j\omega\left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \pi\delta(\omega) + \frac{1}{j\omega} = U(\omega)$$

(9) Shift in frequency domain

$$\mathcal{F}\left[x(t)e^{j\Omega t}\right] = X(\omega - \Omega)$$

proof:

$$\begin{aligned}
\mathcal{F}^{-1}\left[X(\omega-\Omega)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\Omega)e^{j\omega t}d\omega \\
&\quad (\text{let } \omega-\Omega=\omega^{'},\ \ \text{then } \omega=\omega^{'}-\Omega,\ \ \text{and } d\omega=d\omega^{'}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega^{'})e^{j(\omega^{'}+\Omega)t}d\omega^{'} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega^{'})e^{j\omega^{'}t}\cdot e^{j\Omega t}d\omega^{'} \\
&= e^{j\Omega t}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega^{'})e^{j\omega^{'}t}d\omega^{'} \\
&\triangleq x(t)\cdot e^{j\Omega t}
\end{aligned}$$

(10) Modulation

$$\mathcal{F}\left[x(t)y(t)\right] = \frac{1}{2\pi}X(\omega) * Y(\omega)$$

proof: Consider the inverse Fourier transform of the RHS:

$$\begin{aligned}
\mathcal{F}^{-1}\left[\frac{1}{2\pi}X(\omega) * Y(\omega)\right] &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}X(\omega) * Y(\omega)\right\}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\alpha)Y(\omega-\alpha)d\alpha\right\}e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}X(\alpha)\left\{\int_{-\infty}^{\infty}Y(\omega-\alpha)e^{j\omega t}d\omega\right\}d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\alpha)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}Y(\omega-\alpha)e^{j\omega t}d\omega\right\}d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}X(\alpha)y(t)e^{j\alpha t}d\alpha \quad \text{(by property (9))} \\
&= \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\alpha)e^{j\alpha t}d\alpha\right\}\cdot y(t) \\
&= x(t)y(t)
\end{aligned}$$

i.e., $x(t)y(t)$ and $\frac{1}{2\pi}X(\omega) * Y(\omega)$ are Fourier transform pairs!!!

Example 4.4

AM(amplitude modulation) to be discussed in later chapter!

Remark: Refer the Fourier transform properties (p.328) along with Fourier series properties (p.206), and typical Fourier transform pairs (p.329) in the main reference!!!

Examples of Fourier Transforms: typical F.T.

Example 4.5

The rectangular function: $\mathrm{rect}(t)$

$$\mathrm{rect}(t) \triangleq \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Figure 4.21: The rectangular function, $u(t)$.

Solution:

Figure 4.22: The F.T. of the rectangular function, $U(\omega)$.

Example 4.6

Find the Fourier transform of sinc(t).

Solution:

Figure 4.23: The F.T. of the sinc function.

Example 4.7

The triangular function: $\mathrm{tri}(t)$

$$\mathrm{tri}(t) \triangleq \begin{cases} 1 - |t|, & |t| \leq 1 \\ \\ 0 & \text{elsewhere} \end{cases}$$

Figure 4.24: The triangular function.

Solution:

Figure 4.25: The F.T. of the triangular function.

note:

1. $1 - \cos(2\theta) = 2\sin^2(\theta)$
2. The Fourier transforms of $\mathrm{rect}(t)$ and $\mathrm{tri}(t)$ are *pure real*, why?

another easier way:

Notice that

$$\mathrm{tri}(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$$

Figure 4.26: $\mathrm{tri}(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$

proof: assignment

Therefore, we have

$$\begin{aligned}\mathcal{F}[\mathrm{tri}(t)] &= \mathcal{F}[\mathrm{rect}(t)] \cdot \mathcal{F}[\mathrm{rect}(t)] \\ &= \mathcal{F}^2[\mathrm{rect}(t)] \\ &= \mathrm{sinc}^2(f) = \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)\end{aligned}$$

Example 4.8

The symmetrical exponential function

$$x(t) = e^{-a|t|} \quad \text{where } a > 0$$

Figure 4.27: The symmetrical exponential function.

Solution:

.

Figure 4.28: The F.T. of the symmetrical exponential function(:lowpass signal).

Example 4.9

The cosine function

$$x(t) = \cos(\omega_0 t)$$

Figure 4.29: The cosine function.

Solution:

Figure 4.30: The F.T. of the cosine function.

note:

(a) The cosine function is a singular function, and/or it is a periodic signal as well, but its Fourier transform exists.

(b) $\mathcal{F}[x(t)]$ can be obtained in another way as follows:

REMARK:

Relationship between (sampled version of) F.T. and F.S. coefficient for periodic signals

Given a periodic signal $x(t)$:

Figure 4.31: A periodic signal.

Then, if we calculate the F.T. of $x(t)$, we get;

$$\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} C_x(k)e^{j\omega_0 kt} \right) e^{-j\omega t}dt \\
&= \sum_{k=-\infty}^{\infty} C_x(k) \int_{-\infty}^{\infty} e^{-j(\omega - k\omega_0)t}dt \\
&= \sum_{k=-\infty}^{\infty} C_x(k) 2\pi\delta(\omega - k\omega_0) \\
& \quad \left(\delta(\omega - k\omega_0) = 1 \text{ only if } \omega = k\omega_0 \;\rightarrow\; k = \frac{\omega}{\omega_0} \text{ :integer} \right) \\
&= 2\pi C_x\left(\frac{\omega}{\omega_0}\right)
\end{aligned}$$

where $\frac{\omega}{\omega_0} = k$ must be an integer, which means it is only valid when $\omega = k\omega_0$. Also, notice that since integer k is involved, the delta function $\delta(\omega - k\omega_0)$ in above equation is the Kronecker delta function defined as follows:

$$\delta(k) \triangleq \begin{cases} 1, & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the relationship between F.T. and F.S. coefficient for periodic signals can be summarized as follows:

$$X(\omega)|_{\omega = k\omega_0} = 2\pi C_x(k)$$

(cf.) If $T_0 = 2\pi$(sec), then $X(k) = 2\pi C_x(k)$!!!

e.g.:

For $x(t) = \cos(\omega_0 t)$, since it can be represented as follows(using Euler's formula):

$$\begin{aligned} x(t) &= \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} C_x(k) e^{jk\omega_o t} \end{aligned}$$

where the F.S.coefficient of $x(t)$ is given as follows:

$$C_x(k) = \begin{cases} \frac{1}{2}, & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Figure 4.32: F.S. coefficient of the cosine signal.

Assignment:
Verify that the relationship $X(\omega)|_{\omega = k\omega_0} = 2\pi C_x(k)$ holds in this example!!!

Example 4.10

The sine function

$$x(t) = \sin(\omega_0 t)$$

Figure 4.33: The sine signal.

Solution:

Figure 4.34: The F.T. of the sine signal.

(cf)

Figure 4.35: The F.T.'s of the cosine and the sine signals.

e.g.:

For $x(t) = \sin(\omega_0 t)$, since it can be represented as follows(using Euler's formula):

$$\begin{aligned} x(t) &= \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} \\ &= \sum_{k=-\infty}^{\infty} C_x(k) e^{jk\omega_o t} \end{aligned}$$

the F.S.coefficient of $x(t)$ is given as follows:

$$C_x(k) = \begin{cases} \frac{1}{2j}, & k = 1 \\ -\frac{1}{2j}, & k = -1 \\ 0 & \text{otherwise} \end{cases}$$

Using the relationship $X(\omega)|_{\omega=k\omega_0} = 2\pi C_x(k)$, we can readily get the F.T. of $\sin(\omega_0 t)$ as:

$$\begin{aligned} X(\omega) &= 2\pi\frac{1}{2j}\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\} \\ &= j\pi\{\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\} \end{aligned}$$

Comparison between F.S. and F.T.: Summary

	F.S.	F.T.
Time	$x(t) = \sum_{k=-\infty}^{\infty} C_x(k) e^{j\frac{2\pi kt}{T}}$ (i) continuous (ii) periodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ (i) continuous (ii) non-periodic
Frequency	$C_x(k) = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}dt$ (i) discrete in k (ii) non-periodic	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ (i) continuous in ω (ii) non-periodic

Example 4.11

(1) The *time scaled* rectangular function, whose frequency distribution must be analyzed via F.T.

$$x(t) = \mathrm{rect}(\frac{1}{2}t) \triangleq \begin{cases} 1, & |t| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Figure 4.36: The *time scaled* rectangular function.

Solution:

Figure 4.37: The F.T. of the *time scaled* rectangular function.

(2) Construct a periodic signal $\tilde{x}(t)$ from the *time scaled* rectangular function $x(t)$ with period of $T = 4$(sec), whose frequency distribution must be analyzed via F.S. coefficients.

Figure 4.38: The *time scaled* rectangular function train($T = 4$).

Solution:

Figure 4.39: The F.S. coefficients of the *time scaled* rectangular function train($T = 4$).

Question: What is the (sampled version of) F.T. for the above periodic signal $\tilde{x}(t)$?

Answer:

REMARK: Relationship between the F.T. of a finite duration, i.e. non-periodic, signal $x(t)$ and the F.S. coefficient of periodic signal $\tilde{x}(t)$ which is constructed from $x(t)$

Consider a *finite duration* signal $x(t)$ $\ni$:

Figure 4.40: A finite duration signal $x(t)$.

$$X(\omega) = \int_{t_1}^{t_2} x(t)e^{-j\omega t}dt \tag{4.9}$$

Construct a periodic signal $\tilde{x}(t)$ with an arbitrary period of T:

Figure 4.41: A periodic signal $\tilde{x}(t)$.

Then,

$$\begin{aligned} C_{\tilde{x}}(k) &= \frac{1}{T}\int_{t_1}^{t_2} x(t)e^{-j\frac{2\pi kt}{T}}dt \\ &= \frac{1}{T}\int_{t_1}^{t_2} x(t)e^{-jk\omega_0 t}dt \end{aligned} \tag{4.10}$$

where $\omega_0 = \frac{2\pi}{T}$(rad/sec) is the fundamental frequency of $\tilde{x}(t)$.

From (4.9) and (4.10), we have:

$$\begin{aligned} X(\omega)\big|_{\omega=k\omega_0} &= \int_{t_1}^{t_2} x(t)e^{-jk\omega_0 t}dt \\ &= T\cdot C_{\tilde{x}}(k) \end{aligned}$$

Therefore, the relationship between the F.T. of a finite duration(non-periodic) signal $x(t)$ and the F.S. coefficient of periodic signal $\tilde{x}(t)$ constructed from $x(t)$ is in the following form:

$$X(\omega)\big|_{\omega=k\omega_0} = T\cdot C_{\tilde{x}}(k) \tag{4.11}$$

Caution: Do not be confused with the relationship below between the F.S. coefficient and the F.T. of a periodic signal $x(t)$, discussed earlier!!!

$$X(\omega)\big|_{\omega=k\omega_0} = 2\pi\cdot C_x(k)$$

Example 4.12

Take the signals $x(t)$ and $\tilde{x}(t)$ in example 4.11, and check out the relationship in (4.11).

Solution:

4.3 Analysis of LTI system using Fourier Transform

(1) We are given an LTI system as follows:

Figure 4.42: An LTI system

where the impulse response $h(t)$ of the system is:

$$h(t) = e^{-at}u(t), \quad a > 0$$

and thus the transfer function $H(\omega)$ of the system is obtained as:

$$H(\omega) = \frac{1}{a + j\omega}$$

Figure 4.43: The impulse response $h(t)$

1. Suppose $x(t) = \delta(t)$.

   Then, since $h(t)$ is the impulse response, we expect $y(t) = h(t)$.

   (a) Time domain analysis:

$$\begin{aligned} y(t) = h(t) * x(t) &= h(t) * \delta(t) \\ &= \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau \\ &= h(t) \quad \text{(by sifting property of } \delta(t)) \end{aligned}$$

   or

Figure 4.44: The convolution $\delta(t) * h(t)$.

   (i) $t < 0$:

$$y(t) = 0$$

   (ii) $t \geq 0$:

$$\begin{aligned} y(t) &= \int_{0}^{\infty} e^{-a\tau}\delta(t-\tau)d\tau = e^{-at} \\ & \text{(by sifting property of } \delta(t)) \end{aligned}$$

   Therefore, we have $y(t) = e^{-at}u(t) = h(t)$ as expected!!!

   (b) Frequency domain analysis:

$$\begin{aligned} Y(\omega) &= H(\omega) \cdot X(\omega) \quad \text{where } X(\omega) = \mathcal{F}[\delta(t)] = 1 \\ &= \frac{1}{a + j\omega} \cdot 1 \\ &= \frac{1}{a + j\omega} \end{aligned}$$

   Therefore,

$$y(t) = \mathcal{F}^{-1}\left[\frac{1}{a + j\omega}\right] = e^{-at}u(t) = h(t)$$

2. Suppose $x(t) = u(t)$.

(a) Time domain analysis:

$$y(t) = h(t) * x(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

Figure 4.45: The convolution $u(t) * h(t)$.

i) $t < 0$:

$$y(t) = 0$$

(ii) $t \geq 0$:

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau} d\tau \\ &= \left[\frac{e^{-a\tau}}{-a}\right]_0^t \\ &= \frac{1}{a}\{1 - e^{-at}\} \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$

Figure 4.46: The input $x(t)$ and output $y(t)$.

note: Notice that high frequency portions of $x(t)$ have been cut off in $y(t)$, i.e. $h(t)$ is a LPF!!!

(b) Frequency domain analysis:

$$
\begin{aligned}
Y(\omega) = H(\omega)X(\omega) &= \frac{1}{a+j\omega}\left\{\pi\delta(\omega) + \frac{1}{j\omega}\right\} \\
&= \frac{\pi}{a}\delta(\omega) + \frac{1}{j\omega}\frac{1}{a+j\omega} \\
&= \frac{\pi}{a}\delta(\omega) + \frac{1}{a}\left(\frac{1}{j\omega} - \frac{1}{a+j\omega}\right) \quad \text{(partial fraction)} \\
&= \frac{1}{a}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) - \frac{1}{a}\frac{1}{a+j\omega}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y(t) &= \mathcal{F}^{-1}\left[Y(\omega)\right] = \frac{1}{a}u(t) - \frac{1}{a}e^{-at}u(t) \\
&= \frac{1}{a}\left(1 - e^{-at}\right)u(t)
\end{aligned}
$$

(2) Consider another LTI system as follows:

Figure 4.47: Another LTI system

where the input $x(t)$ and the impulse response $h(t)$ of the system are given:

$$x(t) = h(t) = \mathrm{sinc}(t)$$

and thus the input spectrum and the transfer function $H(\omega)$ of the system are obtained as:

$$X(\omega) = H(\omega) = \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

(cf.) The Fourier transform of $\mathrm{sinc}(t)$

Let $x(t) = \mathrm{sinc}(t)$, then we have:

$$\begin{aligned}
& \mathrm{rect}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \\
\Longrightarrow\ & \mathrm{sinc}\left(\frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} 2\pi\mathrm{rect}(\omega) \quad \text{(by duality)} \\
\Longrightarrow\ & \mathrm{sinc}\left(2\pi \cdot \frac{t}{2\pi}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} \cdot 2\pi\mathrm{rect}\left(\frac{\omega}{2\pi}\right) \quad \text{(by time scaling)}
\end{aligned}$$

Therefore,

$$X(\omega) = \mathcal{F}[\mathrm{sinc}(t)] = \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

1. Time domain analysis:

$$
\begin{aligned}
y(t) = h(t) * x(t) &= \mathrm{sinc}(t) * \mathrm{sinc}(t) \\
&= \int_{-\infty}^{\infty} \mathrm{sinc}(\tau)\mathrm{sinc}(t-\tau)d\tau \\
&\quad : \text{very difficult to compute! Try it!!! (assignment)}
\end{aligned}
$$

2. Frequency domain analysis:

$$
Y(\omega) = H(\omega)X(\omega) = \mathrm{rect}^2\left(\frac{\omega}{2\pi}\right) = \mathrm{rect}\left(\frac{\omega}{2\pi}\right)
$$

Figure 4.48: The output spectrum $Y(\omega)$.

Therefore,

$$
y(t) = \mathcal{F}^{-1}[Y(\omega)] = \mathcal{F}^{-1}\left[\mathrm{rect}\left(\frac{\omega}{2\pi}\right)\right] = \mathrm{sinc}(t)
$$

(3) Periodic input to an LTI system:

Suppose the input $x(t)$ to an LTI system is a periodic signal, whereas the impulse response $h(t)$ is non-periodic, then the output signal $y(t)$ of the system is also periodic with the same period(T) of the input:

Figure 4.49: An LTI system with periodic input signal

Objective: Express the F.S.coefficient $C_y(k)$ in terms of the input F.S. coefficient $C_x(k)$ and the transfer function $H(\omega)$ in frequency domain!!!

$$\begin{aligned}
y(t) = h(t) * x(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) \sum_{k=-\infty}^{\infty} C_x(k)e^{j\frac{2\pi k}{T}(t-\tau)}d\tau \\
&= \int_{-\infty}^{\infty} h(\tau) \sum_{k=-\infty}^{\infty} C_x(k)e^{j\frac{2\pi k}{T}t}e^{-j\frac{2\pi k}{T}\tau}d\tau \\
&= \sum_{k=-\infty}^{\infty} C_x(k)e^{j\frac{2\pi k}{T}t} \int_{-\infty}^{\infty} h(\tau)e^{-j\frac{2\pi k}{T}\tau}d\tau \\
&\quad \text{where integration=} \ H(\omega)|_{\omega=\frac{2\pi k}{T}} = H\left(\frac{2\pi k}{T}\right) \\
&= \sum_{k=-\infty}^{\infty} H\left(\frac{2\pi k}{T}\right) \cdot C_x(k)e^{j\frac{2\pi k}{T}t} \\
&\triangleq \sum_{k=-\infty}^{\infty} C_y(k)e^{j\frac{2\pi k}{T}t}
\end{aligned}$$

Therefore,

$$C_y(k) = H\left(\frac{2\pi k}{T}\right) \cdot C_x(k)$$

$$\Downarrow \text{synthesis}$$

$$y(t) = \sum_{k=-\infty}^{\infty} C_y(k)e^{j\frac{2\pi k}{T}t}$$

Another way of derivation:

From the input/output relation of an LTI system in frequency domain:

$$Y(\omega) = H(\omega)X(\omega)$$

We have the following equation which is the sampled version of the above expression at every harmonic frequencies:

$$Y(\omega)\big|_{\omega=\frac{2\pi k}{T}} = H(\omega)X(\omega)\big|_{\omega=\frac{2\pi k}{T}}$$

Then, applying the relation of F.T. and F.S. coefficient for periodic signals, we have

$$2\pi C_y(k) = H\left(\frac{2\pi k}{T}\right) \cdot 2\pi C_x(k)$$

Therefore,

$$C_y(k) = H\left(\frac{2\pi k}{T}\right) \cdot C_x(k)$$

(e.g.) Let $T = 2\pi$, then

$$C_y(k) = H(k) \cdot C_x(k)$$