Contents

Chapter 5

PRACTICAL APPLICATIONS

5.1 Filter

Definition 5.1 A *filter* is an LTI system which passes only *selective portions* of frequency components in the input signal.

Figure 5.1: A filter as an LTI system.

where the input and output spectra can be put into polar forms as:

$$X(\omega) = |X(\omega)|e^{j\Phi_X(\omega)}$$

$$H(\omega) = |H(\omega)|e^{j\Phi_H(\omega)}$$

Then, the output spectrum becomes:

$$\begin{aligned} Y(\omega) = H(\omega)X(\omega) &= |H(\omega)|e^{j\Phi_H(\omega)} \cdot |X(\omega)|e^{j\Phi_X(\omega)} \\ &= |H(\omega)| \cdot |X(\omega)|e^{j(\Phi_H(\omega)+\Phi_X(\omega))} \\ &= |Y(\omega)|e^{j\Phi_Y(\omega)} \end{aligned}$$

Therefore,

$$\begin{aligned} |Y(\omega)| &= |H(\omega)| \cdot |X(\omega)| \quad \text{(magnitude spectrum: \textit{multiplicative})} \\ \Phi_Y(\omega) &= \Phi_H(\omega) + \Phi_X(\omega) \quad \text{(phase spectrum: \textit{additive})} \end{aligned}$$

Example 5.1

The transfer function of a filter is given by: [1]

$$H(\omega) = \frac{1}{a + j\omega} \qquad \text{where } a > 0$$

Then,

$$\begin{aligned} H(\omega) &= \frac{1}{a + j\omega} \\ &= \frac{1}{a^2 + \omega^2}(a - j\omega) \end{aligned}$$

and

$$\begin{aligned} |H(\omega)| &= \frac{1}{\sqrt{a^2 + \omega^2}} \qquad \text{(magnitude)} \\ \Phi_H(\omega) &= \arctan\left(\frac{-\omega}{a}\right) \qquad \text{(phase)} \end{aligned}$$

Figure 5.2: The vector representation of $H(\omega)$.

Figure 5.3: The magnitude and phase of $H(\omega)$.

note:

1. Only low frequency components of $x(t)$ are passed : **LPF**
2. The phase of output $y(t)$ is behind that of input $x(t)$: **Lag(or Delay) Filter**

[1] We already have discussed this type of LTI system before, whose impulse response is $h(t) = e^{-at}u(t)$ and it turned out to be a LPF

Example 5.2

The transfer function of a filter is given by:

$$H(\omega) = a + j\omega(\cdot 1)$$

Then, its impulse response becomes: [2]

$$h(t) = a\delta(t) + \frac{d}{dt}\left\{\delta(t)\right\}$$

and

$$\begin{aligned} |H(\omega)| &= \sqrt{a^2 + \omega^2} \quad \text{(magnitude)} \\ \Phi_H(\omega) &= \arctan\left(\frac{\omega}{a}\right) \quad \text{(phase)} \end{aligned}$$

Figure 5.4: The vector representation of $H(\omega)$.

Figure 5.5: The magnitude and phase of $H(\omega)$.

note:

1. Only high frequency components of $x(t)$ are passed : **HPF**
2. The phase of output $y(t)$ is ahead that of input $x(t)$: **Lead Filter**

[2] This type of filter is physically impossible, i.e. unrealizable, since it requires infinite energy!!!

(cf.) Cascade of filters:

Figure 5.6: Cascade of filters.

$$H(\omega) = \prod_{n=1}^{N} H_n(\omega)$$

and, therefore

$$\begin{aligned} |H(\omega)| &= \prod_{n=1}^{N} |H_n(\omega)| \quad \text{(magnitude)} \\ \Phi_H(\omega) &= \sum_{n=1}^{N} \Phi_{H_n}(\omega) \quad \text{(phase)} \end{aligned}$$

proof: assignment

5.1.1 Ideal Low Pass Filter

The charactersitics of an ideal LPF are given as follows:

(i) magnitude:

Figure 5.7: The magnitude spectrum of an ideal LPF.

(ii) phase: *zero phase*

Figure 5.8: The phase spectrum of an ideal LPF.

meaning: No amplification or attenuation within passband, and no delay in the output signal!!!

Therefore,

$$\begin{aligned} H(\omega) &= |H(\omega)|e^{j\Phi_H(\omega)} \\ &= |H(\omega)| \\ &= \text{rect}\left(\frac{\omega}{2\pi}\right) = \text{rect}(f) \end{aligned}$$

and the impulse response $h(t)$ of an ideal LPF is a sinc function, i.e.

$$h(t) = \mathcal{F}^{-1}\left[\text{rect}(f)\right] = \text{sinc}(t)$$

Figure 5.9: Impulse response of an ideal LPF.

NOTE: Notice that $h(t)$ is NOT a *causal* system (i.e., $h(t) \neq 0$ for $t < 0$), and to make it a causal(or physically realizable: real time) system, we shift $h(t)$ for a sufficient amount of time t_0(i.e. $h_c(t) = h(t - t_0)$.)

Figure 5.10: Impulse response of the causal LPF.

where $h_c(t) \simeq 0$ for $t < 0$.

Therefore,

$$\begin{aligned} H_c(\omega) &= \mathcal{F}\left[h(t - t_0)\right] \\ &= H(\omega)e^{-j\omega t_0} \\ &= \text{rect}\left(\frac{\omega}{2\pi}\right)e^{-j\omega t_0} \\ &\triangleq |H_c(\omega)|e^{j\Phi_{Hc}(\omega)} \end{aligned}$$

and

$$\begin{aligned} |H_c(\omega)| &= \text{rect}\left(\frac{\omega}{2\pi}\right) \quad \text{(magnitude)} \\ \Phi_{H_c}(\omega) &= -\omega t_0 \quad \text{(phase)} \end{aligned}$$

Figure 5.11: The magnitude spectrum of causal LPF.

Figure 5.12: The phase spectrum of causal ideal LPF.: **LINEAR PHASE!!!**

NOTE:
If a filter has a *linear phase* characteristics, there ∃ a delay, but NO distortion at the output signal!!!

(e.g.)

Suppose the input to a filter is as follows:

$$x(t) = \sum_{i=1}^{N} \cos(\omega_i t + \theta)$$

where frequencies $\{\omega_i\}_{i=1}^{N}$ are within the passband of the filter.

Then, since the magnitude is *multiplicative* and the phase is *additive*, the output signal will be:

$$y(t) = \sum_{i=1}^{N} \cos\left(\omega_i t + \theta + \Phi_{H_c}(\omega)\right)$$

Therefore, depending on the phase characteristics $\Phi_{H_c}(\omega)$ of the filter, we have:

(i) Linear phase:

$$y(t) = \sum_{i=1}^{N} \cos\left(\omega_i t + \theta - \omega_i t_0\right) = \sum_{i=1}^{N} \cos\left(\omega_i (t - t_0) + \theta\right)$$

(ii) Non-linear phase:

$$y(t) = \sum_{i=1}^{N} \cos\left(\omega_i t + \theta + f(\omega_i)\right) \quad \text{:distortion due to } f(\omega_i)$$

5.1.2 Ideal Band Pass Filter

The charactersitics of an ideal BPF are given as follows:

(i) magnitude:

Figure 5.13: The magnitude spectrum of an ideal BPF ($\omega = \beta$: center frequency).

(ii) phase: *zero phase*

Figure 5.14: The phase spectrum of an ideal BPF.

Therefore,

$$\begin{aligned} H(\omega) &= |H(\omega)|e^{j\Phi_H(\omega)} \\ &= |H(\omega)| \\ &= \text{rect}\left(\frac{\omega - \beta}{2\pi}\right) + \text{rect}\left(\frac{\omega + \beta}{2\pi}\right) \end{aligned}$$

and the impulse response $h(t)$ of an ideal BPF becomes:

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\left[\text{rect}\left(\frac{\omega - \beta}{2\pi}\right) + \text{rect}\left(\frac{\omega + \beta}{2\pi}\right)\right] \\ &= \text{sinc}(t) \cdot e^{-j\beta t} + \text{sinc}(t) \cdot e^{j\beta t} \quad \text{(by frequency shift)} \\ &= \text{sinc}(t) \cdot \left\{e^{-j\beta t} + e^{j\beta t}\right\} \\ &= 2\text{sinc}(t) \cdot \cos(\beta t) \end{aligned}$$

Figure 5.15: Impulse response of an ideal BPF.

NOTE:
Notice that $h(t)$ is again NOT a *causal* system (i.e., $h(t) \neq 0$ for $t < 0$), and to make it a causal(or physically realizable: real time) system, we shift $h(t)$ for a sufficient amount of time t_0(i.e. $h_c(t) = h(t - t_0)$.)

Figure 5.16: Impulse response of the causal BPF.

where $h_c(t) \simeq 0$ for $t < 0$.

Therefore,

$$\begin{aligned} H_c(\omega) &= \mathcal{F}\left[h(t-t_0)\right] \\ &= H(\omega)e^{-j\omega t_0} \\ &= \left\{\text{rect}\left(\frac{\omega-\beta}{2\pi}\right) + \text{rect}\left(\frac{\omega+\beta}{2\pi}\right)\right\} e^{-j\omega t_0} \\ &\triangleq |H_c(\omega)|e^{j\Phi_{Hc}(\omega)} \end{aligned}$$

and

$$\begin{aligned} |H_c(\omega)| &= \text{rect}\left(\frac{\omega-\beta}{2\pi}\right) + \text{rect}\left(\frac{\omega+\beta}{2\pi}\right) \qquad \text{(magnitude)} \\ \Phi_{H_c}(\omega) &= -\omega t_0 \qquad \text{(phase)} \end{aligned}$$

Figure 5.17: Magnitude & phase spectra of causal ideal BPF: **LINEAR PHASE!!!**.

5.2 Signal Moduation and Demodulation

AMSC: Amplitude Modulation with Suppressed Carrier

Let's define the following notations:

(i) $a(t)$: modulating signal or message, i.e. the signal that we want to send out [3]

(ii) $A(\omega) \triangleq \mathcal{F}[a(t)]$

(iii) $c(t)$: carrier signal

(iv) ω_0: carrier frequency

(v) $y(t)$ modulated signal

(vi) $r(t)$: partially demodulated signal

(vii) $d(t)$: demodulated signal

(1) Block Diagram

Figure 5.18: Block diagram of AMSC system.

[3] Mostly, audible signals such as voice, music etc..

(2) Why Modulation?

1. In communication systems, most signals propagate through the atmosphere, and

   (a) Audible frequency [4] : signals are rapidly attenuated: *short range*

   (b) Higher frequency: signals propagate over longer distance: *long range*

   Therefore, we need a carrier with higher frequency to carry the message(audible) signals to the destination, and this is done via *modulation*

2. It is well known that the antenna size(ℓ) is proportional to the wavelength(λ) of the signal, i.e.

   $$\ell \geq \frac{\lambda}{10}$$

   e.g.

   if the signal frequency is 1(KHz), then $\lambda = c/f = 300$(Km), and $\ell \geq 30$(Km)!!!

3. Interference: to avoid interference among messages, we need to assign different frequency bands to different messages.

[4]The frequency range is from 10(Hz) to 20(KHz)

(3) Analysis of Modulation/Demodulation: time/frequency domains

(3-1) Modulation

From the block diagram, the modulated signal $y(t)$ is:

$$\begin{aligned} y(t) &= a(t) \cdot c(t) \\ &= a(t) \cos(\omega_0 t) \end{aligned} \tag{5.1}$$

Take the Fourier transform of both sides, then

$$\begin{aligned} Y(\omega) &= \frac{1}{2\pi} \{A(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]\} \\ &= \frac{1}{2} \{A(\omega - \omega_0) + A(\omega + \omega_0)\} \end{aligned} \tag{5.2}$$

where we used the *sifting property* of $\delta(t) \ni$:

$$\begin{aligned} \delta(t - t_0) * x(t) &= \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau - t_0) x(t - t_0) d\tau \\ &= x(t - t_0) \end{aligned}$$

Figure 5.19: Sifting process by $\delta(t - t_0)$.

or

we can use other property of F.T. to derive $Y(\omega)$ as:

$$\begin{aligned} y(t) &= a(t) \cos(\omega_0 t) \\ &= \frac{1}{2} a(t) \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \end{aligned}$$

Therefore, by the “frequency shift” property of F.T., we get

$$Y(\omega) = \frac{1}{2} \{A(\omega - \omega_0) + A(\omega + \omega_0)\}$$

(3-2) Demodulation

In the block diagram, the partially demodulated signal $r(t)$ at the receiver is expressed as:

$$\begin{aligned} r(t) &= y(t) \cdot c(t) \\ &= y(t) \cos(\omega_0 t) \end{aligned} \tag{5.3}$$

Take the Fourier transform of both sides, then

$$\begin{aligned} R(\omega) &= \frac{1}{2\pi} \{Y(\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]\} \\ &= \frac{1}{2} \{Y(\omega - \omega_0) + Y(\omega + \omega_0)\} \\ &= \frac{1}{2} \left\{ \frac{1}{2} [A(\omega - 2\omega_0) + A(\omega)] + \frac{1}{2} [A(\omega) + A(\omega + 2\omega_0)] \right\} \\ &= \frac{1}{2} A(\omega) + \frac{1}{4} \{A(\omega - 2\omega_0) + A(\omega + 2\omega_0)\} \end{aligned} \tag{5.4}$$

or

we can directly derive $R(\omega)$ as:

$$\begin{aligned} r(t) &= y(t) \cos(\omega_0 t) \\ &= a(t) \cos^2(\omega_0 t) \\ &= \frac{1}{2} a(t) \left(1 + \cos(2\omega_0 t)\right) \\ &= \frac{1}{2} a(t) + \frac{1}{4} a(t) \left\{ e^{j2\omega_0 t} + e^{-j2\omega_0 t} \right\} \end{aligned}$$

Therefore, by the “linearity” and “frequency shift” properties of F.T., we get

$$R(\omega) = \frac{1}{2} A(\omega) + \frac{1}{4} \{A(\omega - 2\omega_0) + A(\omega + 2\omega_0)\}$$

Based on the above analyses, for each step of modulation/demodulation procedure, the signals and their spectra will change as follows:

Time domain | Frequency domain

Figure 5.20: Signal propagation in AMSC system.

After we get $r(t)$, we apply a LPF with bandwidth=$W' > W$, where $W < W' < \omega_0$, and an amplifier with gain of 2, then we can recover the original message $a(t)$ from $r(t)$, i.e.

Figure 5.21: Ideal LPF with gain of 2 in AMSC system.

Remarks:

(1) AM vs. FM[5]

1. AM: message information is in the *amplitude* of the carrier
2. FM: message information is in the *frequency* of the carrier

e.g.

In AM, the amplitude of the carrier $c(t) = \cos(\omega_0 t)$ varies according to the message $a(t)$, whereas in FM, the frequency of the carrier $c(t) = \cos(\omega_0 t)$ varies according to the message $a(t)$.

Figure 5.22: AM versus FM

(2) Variations of AM [6]

1. AM : $y(t) = \{a(t) + 1\} \cos(\omega_0 t)$: transmission power increase
2. DSB(double side band) modulation
3. DSBSC(DSB with suppressed carrier)
4. SSB(single side band) modulation
5. SSBSC(SSB with suppressed carrier)

[5]FM: Frequency Modulation

[6]More detailed coverage will be discussed in communication theory class.