Contents

Chapter 6

CORRELATION FUNCTION AND SPECTRAL DENSITY

6.1 Periodic Signals

6.1.1 Autocorrelation function

Definition 6.1 The autocorrelation function for a periodic signal $x(t)$ with period T is defined as:

$$\begin{aligned} R_{xx}(\tau) &\triangleq \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt \\ &= x(t)\otimes x(t) \end{aligned}$$

Note:

1. $R_{xx}(\tau)$ provides the *quantitative* indication of resemblance(closeness) between $x(t)$ and $x(t+\tau)$ as we vary τ(amount of shift)!!!
2. Dimension of $R_{xx}(\tau)$ is *power* in watts.
3. $\tau = 0$ gives the average power of $x(t)$, i.e. $R_{xx}(0) = \textit{average power}$

(cf.)

$R_{xx}(\tau)$ is also periodic with the same period T of $x(t)$

proof: assignment

Example 6.1

Find the autocorrelation function of $x(t) = \cos(t)$.

Figure 6.1: $\cos(t)$.

Solution:

Figure 6.2: $R_{xx}(\tau)$.

note:

(a) $\cos(A)\cos(B) = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$

(b) $R_{xx}(\tau)$: periodic with period $T = 2\pi$.

Example 6.2

Find the autocorrelation function of $x(t) = \sin(t)$.

Figure 6.3: $\sin(t)$.

Solution:

Figure 6.4: $R_{xx}(\tau)$.

note:

(a) $\sin(A)\sin(B) = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$

(b) $R_{xx}(\tau)$: periodic with period $T = 2\pi$.

(c) $\cos(t) \otimes \cos(t) \equiv \sin(t) \otimes \sin(t)$!!!!!

Graphical Interpretation:

Figure 6.5: Interpretation of autocorrelation function

Note:

1. $\tau = \frac{\pi}{2}$: case when $x(t)$ and $x(t+\tau)$ are *far from resemblance.*
2. $\tau = \pi$: case when $x(t)$ and $x(t+\tau)$ are *closest in opposite sense.*
3. $\tau = 0$: case when $x(t)$ and $x(t+\tau)$ are *exactly the same.*

Example 6.3

Find the autocorrelation function of following $x(t)$.

$$x(t) \triangleq \tilde{\text{rect}}(t) = \begin{cases} \text{rect}(t) & -1 < t \leq 1 \\ 0 & \text{otherwise}, \qquad \text{where } T = 2 \end{cases}$$

Figure 6.6: $x(t)$.

Solution:

Figure 6.7: $R_{xx}(\tau)$.

note: $R_{xx}(\tau)$: periodic with period $T = 2$.

6.1.2 Cross-correlation function

Definition 6.2 The cross-correlation function of two periodic signals $x(t)$ with period T_1 and $y(t)$ with period T_2 is defined as follows:

$$\begin{aligned} R_{xy}(\tau) &\triangleq \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)dt \\ &= x(t) \otimes y(t) \end{aligned}$$

where T is the least common period(LCP) between T_1 and T_2, e.g. if $T_1 = 2\pi$ and $T_2 = 3\pi$, then $T = 6\pi$

Note: $R_{xy}(\tau) \neq R_{yx}(\tau)$!!!!!

Example 6.4

Find the cross-correlation functions $R_{xy}(\tau)$ and $R_{yx}(\tau)$ respectively for $x(t) = \sin(t)$ and $y(t) = \cos(t)$.

Figure 6.8: $x(t)$ and $y(t)$.

Solution:

(i) $R_{xy}(\tau)$:

Figure 6.9: $R_{xy}(\tau)$.

note:

(a) $\sin(A)\cos(B) = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

(b) $R_{xy}(\tau)$: periodic with period $T = \text{LCP}\{T_1, T_2\} = 2\pi$.

(ii) $R_{yx}(\tau)$:

Figure 6.10: $R_{yx}(\tau)$.

note:

(a) $\cos(A)\sin(B) = \frac{1}{2}\{\sin(A+B) - \sin(A-B)\}$

(b) $R_{yx}(\tau)$: periodic with period $T = \text{LCP}\{T_1, T_2\} = 2\pi$.

(c) $R_{xy}(\tau) \neq R_{yx}(\tau)$

Harmonically related sinusoids:

Given:

$$x(t) = \cos(at)$$
$$y(t) = \cos(bt)$$

where $a \neq b$ and b/a is an integer (i.e. $b = ka$). Then $x(t)$ and $y(t)$ are called *harmonically related* sinusoids!!!

The cross-correlation function, then, bewteen $x(t)$ and $y(t)$ is:

$$\begin{aligned} R_{xy}(\tau) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(at)\cos(bt+b\tau)dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}\left\{\cos[(a+b)t+b\tau]+\cos[(a-b)t-b\tau]\right\}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{2}\left\{\cos[(1+k)at+b\tau]+\cos[(1-k)at-b\tau]\right\}dt \\ &= 0 \end{aligned}$$

where $T =$ LCP.

meaning:
We cannot match $x(t)$ and $y(t)$ no matter how we shift one of them!!!
e.g.

Figure 6.11: $R_{xy}(\tau)$ for harmonically related sinusoids.

Likewise,

$$\cos(at) \otimes \sin(bt) = 0$$
$$\sin(at) \otimes \sin(bt) = 0$$

where $b/a =$ integer.

Computing F.S. coefficient using Cross-Correlation

Given a periodic signal $x(t)$ with period 2π, then the trigonometric F.S. representation of $x(t)$ is as follows:

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nt) + b_n \sin(nt)\}$$

Define,

$$R_{kx}(\tau) \triangleq \cos(kt) \otimes x(t) \qquad \text{where } k = 0, 1, 2, \ldots$$

Then

$$R_{kx}(\tau) = \cos(kt) \otimes \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n \cos(nt) + b_n \sin(nt)\} \right)$$

(i) $k = 0$

$$R_{0x}(\tau) = 1 \otimes \frac{a_0}{2} = \frac{1}{2\pi} \int_{2\pi} \frac{a_0}{2} dt = \frac{a_0}{2}$$

$$\longrightarrow a_0 = 2R_{0x}(\tau) \longrightarrow a_0 = 2R_{0x}(0)$$

since $R_{0x}(\tau)$ is independent of τ, i.e. constant

(ii) $k = 1$

$$R_{1x}(\tau) = \cos(t) \otimes \{a_1 \cos(t) + b_1 \sin(t)\} = \frac{a_1}{2} \cos(\tau) + \frac{b_1}{2} \sin(\tau)$$

$$\longrightarrow a_1 = 2R_{1x}(0) \quad \text{and} \quad b_1 = 2R_{1x}(\frac{\pi}{2})$$

(iii) $k = 2$

$$R_{2x}(\tau) = \cos(2t) \otimes \{a_2 \cos(2t) + b_2 \sin(2t)\} = \frac{a_2}{2} \cos(2\tau) + \frac{b_2}{2} \sin(2\tau)$$

$$\longrightarrow a_2 = 2R_{2x}(0) \quad \text{and} \quad b_2 = 2R_{2x}(\frac{\pi}{2 \cdot 2})$$

$$\vdots$$

Therefore, in general, the trigonometric F.S. coefficients can be calculated as:

$$\begin{cases} a_n = 2R_{nx}(0) & \text{for } n = 0, 1, 2, \ldots \\ b_n = 2R_{nx}(\frac{\pi}{2n}) & \text{for } n = 1, 2, \ldots \end{cases}$$

and corresponding complex F.S. coefficients are

$$C_k \triangleq \frac{1}{2}(a_k - jb_k) = R_{kx}(0) - jR_{kx}(\frac{\pi}{2k})$$

6.1.3 Property of autocorrelation function

Property #1:

$$R_{xx}(0) \geq 0$$

proof:

$$\begin{aligned} R_{xx}(0) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+0)dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt \\ &\geq 0 \end{aligned}$$

Property #2:

$$R_{xx}(\tau) = R_{xx}(-\tau) \quad \text{(even function of } \tau)$$

proof:

$$\begin{aligned} \text{RHS} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t-\tau)dt \\ & (Let\ t-\tau = t^{'}) \\ &= \frac{1}{T}\int_{-\frac{T}{2}+\tau}^{\frac{T}{2}+\tau} x(t^{'}+\tau)x(t^{'})dt^{'} \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t^{'})x(t^{'}+\tau)dt^{'} \\ &= R_{xx}(\tau) \\ &= \text{LHS} \end{aligned}$$

Property #3:

$$R_{xx}(0) \geq R_{xx}(\tau) \quad (R_{xx}(0) \text{ is the maximum})$$

proof:

Define

$$y(t) \triangleq x(t) - x(t+\tau)$$

Then,

$$\begin{aligned}
R_{yy}(0) &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y^2(t)dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \{x(t) - x(t+\tau)\}^2\, dt \\
&= \frac{1}{T}\left\{\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t+\tau)dt - 2\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt\right\} \\
&= R_{xx}(0) + R_{xx}(0) - 2R_{xx}(\tau) \\
&\geq 0 \quad \text{(should be)}
\end{aligned}$$

Therefore,

$$2R_{xx}(0) - 2R_{xx}(\tau) \geq 0$$

$$\longrightarrow R_{xx}(0) \geq R_{xx}(\tau) \ \ \forall \tau$$

6.1.4 Property of cross-correlation function

Property #1:

$$R_{xy}(\tau) = R_{yx}(-\tau)$$

proof:

$$\begin{aligned}
\text{LHS} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)dt \\
&\quad (Let\ \ t+\tau = t^{'}) \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t^{'}-\tau)y(t^{'})dt^{'} \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t^{'})x(t^{'}-\tau)dt^{'} \\
&\triangleq R_{yx}(-\tau) \\
&= \text{RHS}
\end{aligned}$$

(cf.) Recall that in previous example, for $x(t) = \sin(t)$ and $y(t) = \cos(t)$, we found that $R_{xy}(\tau) = -\frac{1}{2}\sin(\tau)$ whereas $R_{yx}(\tau) = \frac{1}{2}\sin(\tau) = R_{xy}(-\tau)$!!!

Property #2:

$$x(t+a) \otimes y(t+b) = R_{xy}(\tau + b - a)$$

proof:

$$\begin{aligned}
\text{LHS} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t+a)y(t+b+\tau)dt \\
&\quad (Let\ \ t+a = t^{'}) \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t^{'})y(t^{'}+\tau+b-a)dt^{'} \\
&\triangleq R_{xy}(\tau+b-a) \\
&= \text{RHS}
\end{aligned}$$

where $b-a$ is the difference of the arrival(or delay) time between $x(t)$ and $y(t)$.

Example 6.5

Radar ranging: We want to estimate the *time delay* t_0, i.e. $\hat{t}_0$.

Figure 6.12: Typical active radar system

We usually use a train of pulses, modulated with microwave signal, for $x(t)$ in general:

Figure 6.13: $x(t)$.

Figure 6.14: $R_{xx}(\tau)$.

If we take the cross-correlation between $x(t)$ and $y(t)$, we get

$$R_{xy}(\tau) = x(t) \otimes [\alpha x(t - t_0)] = \alpha R_{xx}(\tau - t_0)$$

Figure 6.15: $R_{xy}(\tau)$.

NOTE:

(a) By detecting the peak location of $R_{xy}(\tau)$ within T, we can estimate the time delay $\hat{t}_0$, i.e.

$$\hat{t}_0 = \text{argmax}_{\tau \in T} R_{xy}(\tau)$$

And the distance L between the radar site and the target can then be estimated as,

$$\hat{L} = \frac{\hat{t}_0}{2} \cdot C$$

where C is the velocity of the signal $x(t)$.

(b) To avoid ambiguity, the following condition should be satisfied, i.e.

$$t_0 \leq T$$

Therefore, the maximum detection range $L_{\max}$ is determined by:

$$L_{\max} = \frac{T}{2} \cdot C \propto T$$

Property #3:

$$x(t) \otimes y^{'}(t) = \frac{d}{d\tau}\left\{R_{xy}(\tau)\right\}$$

proof:

$$\begin{aligned}
\text{LHS} &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y^{'}(t+\tau)dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\frac{d}{dt}\left\{y(t+\tau)\right\}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\frac{d}{d\tau}\left\{y(t+\tau)\right\}dt \\
& \left(\text{since } \frac{d}{dt}\left\{y(t+\tau)\right\} = \frac{d}{d\tau}\left\{y(t+\tau)\right\}\right) \\
&= \frac{d}{d\tau}\left\{\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)y(t+\tau)dt\right\} \quad \text{(by Leibniz rule)} \\
&= \frac{d}{d\tau}\left\{R_{xy}(\tau)\right\} \\
&= \text{RHS}
\end{aligned}$$

Note: This is very useful in radar ranging when the peak location of $R_{xy}(\tau)$ is ambiguous:

Peak detection $\Longrightarrow$ Zero-crossing detection

(e.g.)

(i) $R_{xy}(\tau)$ with clear peak

Figure 6.16: $R_{xy}(\tau)$: clear peak.

(ii) $R_{xy}(\tau)$ with ambiguous peak

Figure 6.17: $R_{xy}(\tau)$: ambiguous peak.

6.1.5 Power spectral density

Definition 6.3 The power spectral densities for periodic signals $x(t)$ and $y(t)$ with period T are defined as:

(1) Auto power spectral density of $x(t)$:

$$\begin{aligned} P_{xx}(k) &\triangleq C_x^*(k) \cdot C_x(k) \\ &= |C_x(k)|^2 \end{aligned}$$

where $C_x(k)$ is the F.S. coefficient of $x(t)$, and $P_{xx}(k)$ represents the distribution of power in $x(t)$ with respect to k(i.e. frequency).

(2) Cross power spectral density between $x(t)$ and $y(t)$:

$$P_{xy}(k) \triangleq C_x^*(k) \cdot C_y(k)$$

where $C_x(k)$ and $C_y(k)$ are the F.S. coefficients of $x(t)$ and $y(t)$ respectively.

Theorem 6.1 The power spectral density of periodic signals is the F.S. coefficient of their correlation functions:

$$P_{xy}(k) = C_R(k)$$

where $C_R(k)$ is the F.S. coefficient of the cross-correlation function $R_{xy}(\tau)$ between $x(t)$ and $y(t)$, i.e.

$$\begin{aligned} R_{xy}(\tau) &= \frac{1}{T}\int_T x(t)y(t+\tau)dt \\ &= \sum_{k=-\infty}^{\infty} C_R(k)e^{j\frac{2\pi k\tau}{T}} \end{aligned}$$

where T=LCP of $x(t)$ and $y(t)$.

Proof:
We must show that

$$C_R(k) = C_x^*(k) \cdot C_y(k) \triangleq P_{xy}(k)$$

$$
\begin{aligned}
\text{LHS} = C_R(k) &\triangleq \frac{1}{T}\int_T R_{xy}(\tau)e^{-j\frac{2\pi k\tau}{T}}d\tau \\
&= \frac{1}{T}\int_T \left\{\frac{1}{T}\int_T x(t)y(t+\tau)dt\right\} e^{-j\frac{2\pi k\tau}{T}}d\tau \\
&= \frac{1}{T}\int_T x(t)\left\{\frac{1}{T}\int_T y(t+\tau)e^{-j\frac{2\pi k\tau}{T}}d\tau\right\}dt \\
&\quad (let \;\; t+\tau = \tau^{'}) \\
&= \frac{1}{T}\int_T x(t)\left\{\frac{1}{T}\int_T y(\tau^{'})e^{-j\frac{2\pi k(\tau^{'}-t)}{T}}d\tau^{'}\right\}dt \\
&= \left\{\frac{1}{T}\int_T x(t)e^{j\frac{2\pi kt}{T}}dt\right\}\cdot\left\{\frac{1}{T}\int_T y(\tau^{'})e^{-j\frac{2\pi k\tau^{'}}{T}}d\tau^{'}\right\} \\
&\quad (\text{assuming } x(t) \text{ is real}) \\
&= \left\{\frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}dt\right\}^{*}\cdot\left\{\frac{1}{T}\int_T y(\tau^{'})e^{-j\frac{2\pi k\tau^{'}}{T}}d\tau^{'}\right\} \\
&\triangleq C_x^*(k)\cdot C_y(k) \\
&= \text{RHS}
\end{aligned}
$$

Reminder: Power spectral density is the F.S. coefficient of the correlation function for periodic signals!!!!!

6.2 Non-periodic Signals

6.2.1 Autocorrelation function

Definition 6.4 The autocorrelation function for a non-periodic signal $x(t)$ is defined as:

$$\begin{aligned} R_{xx}(\tau) &\triangleq \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \\ &= x(t) \otimes x(t) \end{aligned}$$

Note:

1. $R_{xx}(\tau)$ indicates the resemblance(closeness) between $x(t)$ and $x(t+\tau)$ *quantitatively* as we vary τ(amount of shift)!!!
2. Dimension of $R_{xx}(\tau)$ is *energy* in joules.
3. $\tau = 0$ gives the energy of $x(t)$, i.e. $R_{xx}(0) = energy\ of\ x(t)$

(cf.)

It is assumed that $x(t)$ has a finite energy, i.e.,

$$\int_{-\infty}^{\infty} x^2(t)dt < \infty$$

6.2.2 Cross-correlation function

Definition 6.5 Similarly, we define the cross-correlation function between two non-periodic signals $x(t)$ and $y(t)$ as:

$$\begin{aligned} R_{xy}(\tau) &\triangleq \int_{-\infty}^{\infty} x(t)y(t+\tau)dt \\ &= x(t) \otimes y(t) \end{aligned}$$

6.2.3 Property of autocorrelation function

Same as the properties of autocorrelation function for a periodic signal:

Property #1:

$$R_{xx}(0) \geq 0$$

Property #2:

$$R_{xx}(\tau) = R_{xx}(-\tau) \quad \text{(even function of } \tau)$$

Property #3:

$$R_{xx}(0) \geq R_{xx}(\tau) \quad (R_{xx}(0) \text{ is the maximum})$$

proof: assignment

6.2.4 Property of cross-correlation function

Same as the properties of cross-correlation function for periodic $x(t)$ and $y(t)$:

Property #1:

$$R_{xy}(\tau) = R_{yx}(-\tau)$$

Property #2:

$$x(t+a) \otimes y(t+b) = R_{xy}(\tau + b - a)$$

Property #3:

$$x(t) \otimes y^{'}(t) = \frac{d}{d\tau}\left\{R_{xy}(\tau)\right\}$$

proof: assignment

Example 6.6

Find the autocorrelation function of $x(t)$ given below.

$$x(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

Figure 6.18: $x(t)$.

Solution:

Figure 6.19: $R_{xx}(\tau)$.

Example 6.7

Find the autocorrelation function of $x(t)$ given below.

$$x(t) = e^{-at}u(t), \quad a > 0$$

Figure 6.20: $x(t)$ and $x(t+\tau)$ for $\tau < 0$.

Solution:

Figure 6.21: $R_{xx}(\tau)$.

6.2.5 Energy spectral density

Definition 6.6 The energy spectral densities for non-periodic signals $x(t)$ and $y(t)$ are defined as:

(1) Auto energy spectral density of $x(t)$:

$$\begin{aligned} S_{xx}(\omega) &\triangleq X^*(\omega) \cdot X(\omega) \\ &= |X(\omega)|^2 \end{aligned}$$

where $X(\omega)$ is the F.T. of $x(t)$, and $S_{xx}(\omega)$ represents the distribution of energy in $x(t)$ with respect to ω(i.e. frequency).

(2) Cross energy spectral density between $x(t)$ and $y(t)$:

$$S_{xy}(\omega) \triangleq X^*(\omega) \cdot Y(\omega)$$

where $X(\omega)$ and $Y(\omega)$ are the F.T.'s of $x(t)$ and $y(t)$ respectively.

Theorem 6.2 (Wiener-Khinchin Theorem:)
The correlation function and the energy spactral density is a Fourier transform pair, i.e.:

$$\begin{aligned} R_{xx}(\tau) &\overset{\mathcal{F}}{\longleftrightarrow} S_{xx}(\omega) \\ R_{xy}(\tau) &\overset{\mathcal{F}}{\longleftrightarrow} S_{xy}(\omega) \end{aligned}$$

where $x(t)$ and $y(t)$ are assumed to be *real* signals.

Proof:

$$\begin{aligned} \mathcal{F}\left[R_{xy}(\tau)\right] &= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(t) y(t+\tau) dt \right\} e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} x(t) \left\{ \int_{-\infty}^{\infty} y(t+\tau) e^{-j\omega\tau} d\tau \right\} dt \\ &= \int_{-\infty}^{\infty} x(t) \cdot Y(\omega) e^{j\omega t} dt \\ &= \left\{ \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right\} \cdot Y(\omega) \\ & \quad \text{(assuming } x(t) \text{ is real)} \\ &= \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right\}^* \cdot Y(\omega) \\ &= X^*(\omega) Y(\omega) \\ &\triangleq S_{xy}(\omega) \end{aligned}$$

Example 6.8

Find the auto energy spectral density of $x(t)$ discussed in example 6.7.

$$x(t) = e^{-at}u(t), \quad a > 0$$

Figure 6.22: $x(t)$.

Solution:

Figure 6.23: $S_{xx}(\omega)$.

6.2.6 Input/output relation of an LTI system(in terms of energy)

Given an LTI system,

Figure 6.24: LTI system.

where $h(t)$ and $H(\omega)$ are the impulse response and the transfer function of the system respectively.

(1) Output signal:

$$y(t) = h(t) * x(t)$$

$$Y(\omega) = H(\omega)X(\omega)$$

(2) Output auto energy spectral density:

$$\begin{aligned} S_{yy}(\omega) &\triangleq Y^*(\omega)Y(\omega) \\ &= \{H(\omega)X(\omega)\}^* \{H(\omega)X(\omega)\} \\ &= H^*(\omega)X^*(\omega)H(\omega)X(\omega) \\ &= |H(\omega)|^2 \cdot X^*(\omega)X(\omega) \\ &= |H(\omega)|^2 \cdot S_{xx}(\omega) \end{aligned}$$

where $|H(\omega)|^2$ is called the "*energy transfer function*" of the system.

(3) Output autocorrelation function:

Using the Wiener-Khinchin theorem,

$$\begin{aligned} R_{yy}(\tau) &= \mathcal{F}^{-1}\{S_{yy}(\omega)\} \\ &= \mathcal{F}^{-1}\left\{|H(\omega)|^2 \cdot S_{xx}(\omega)\right\} \\ &= \mathcal{F}^{-1}\left\{|H(\omega)|^2\right\} * \mathcal{F}^{-1}\{S_{xx}(\omega)\} \\ &= \mathcal{F}^{-1}\left\{|H(\omega)|^2\right\} * R_{xx}(\tau) \end{aligned}$$

(4) Cross energy spectral density b/w input and output:

$$\begin{aligned} S_{xy}(\omega) &\triangleq X^*(\omega)Y(\omega) \\ &= X^*(\omega)H(\omega)X(\omega) \\ &= H(\omega)\{X^*(\omega)X(\omega)\} \\ &= H(\omega)S_{xx}(\omega) \end{aligned}$$

(5) Cross-correlation function b/w input and output:

Using the Wiener-Khinchin theorem,

$$\begin{aligned} R_{xy}(\tau) &= \mathcal{F}^{-1}\{S_{xy}(\omega)\} \\ &= \mathcal{F}^{-1}\{H(\omega) \cdot S_{xx}(\omega)\} \\ &= \mathcal{F}^{-1}\{H(\omega)\} * \mathcal{F}^{-1}\{S_{xx}(\omega)\} \\ &= h(\tau) * R_{xx}(\tau) \end{aligned}$$

(6) Cross energy spectral density b/w output and input:

$$\begin{aligned} S_{yx}(\omega) &\triangleq Y^*(\omega)X(\omega) \\ &= \{H(\omega)X(\omega)\}^* X(\omega) \\ &= H^*(\omega)X^*(\omega)X(\omega) \\ &= H^*(\omega)S_{xx}(\omega) \end{aligned}$$

note: $S_{yx}(\omega) = Y^*(\omega)X(\omega) = \{X^*(\omega)Y(\omega)\}^* = S^*_{xy}(\omega)$.

(7) Cross-correlation function b/w output and input:

Using the Wiener-Khinchin theorem,

$$\begin{aligned}
R_{yx}(\tau) &= \mathcal{F}^{-1}\{S_{yx}(\omega)\} \\
&= \mathcal{F}^{-1}\{H^*(\omega)\cdot S_{xx}(\omega)\} \\
&= h(-\tau) * R_{xx}(\tau) \quad \text{(assuming } h(\tau) \text{ is real)} \\
&= h(-\tau) * R_{xx}(-\tau) \quad \text{(since } R_{xx}(\tau) \text{ is symmetric)} \\
&= R_{xy}(-\tau)
\end{aligned}$$

Note:

(i)

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

(ii)

$$\begin{aligned}
\mathcal{F}^{-1}[H^*(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H^*(\omega)e^{j\omega\tau}d\omega \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-j\omega\tau}d\omega\right)^* \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega(-\tau)}d\omega\right)^* \\
&= h^*(-\tau) \\
&= h(-\tau) \quad \text{(assuming } h(\tau) \text{ is real)}
\end{aligned}$$

6.3 Parseval's Theorem

Theorem 6.3 The energy in time domain must be equal to the energy in frequency domain, so the energy of a non-periodic signal $x(t)$ can be computed as the integration of $S_{xx}(\omega)$ scaled by $\frac{1}{2\pi}$, i.e.

$$E = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega = R_{xx}(0)$$

Proof:

From the Wiener-Khinchin theorem, we have,

$$R_{xx}(\tau) \triangleq \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \equiv \mathcal{F}^{-1}\left[S_{xx}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega$$

Let $\tau = 0$ at each term of both sides, then

$$R_{xx}(0) = \int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega$$

Q.E.D.

Simailary, for the periodic signals, we have the following Parseval's theorem:

Theorem 6.4 The power in time domain must be equal to the power in frequency domain, so the average power of a periodic signal $x(t)$ can be computed as the sum of $P_{xx}(k)$, i.e.

$$P = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} P_{xx}(k) = R_{xx}(0)$$

Proof:

$$\begin{aligned}
\text{LHS} &= \frac{1}{T}\int_T |x(t)|^2 dt \\
&= \frac{1}{T}\int_T x^*(t)x(t)dt \\
&= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} C_x^*(k)e^{-j\frac{2\pi kt}{T}}\right) x(t)dt \\
&= \sum_{k=-\infty}^{\infty} C_x^*(k)\cdot\frac{1}{T}\int_T x(t)e^{-j\frac{2\pi kt}{T}}dt \\
&= \sum_{k=-\infty}^{\infty} C_x^*(k)\cdot C_x(k) = \sum_{k=-\infty}^{\infty} |C_x(k)|^2 = \sum_{k=-\infty}^{\infty} P_{xx}(k) = \text{RHS}
\end{aligned}$$

Q.E.D.

note:
If $x(t)$ is a *real signal*, it is simpler to prove the theorem:

proof:

$$\begin{aligned}
\text{LHS} &= \frac{1}{T}\int_T x^2(t)dt \\
&= \frac{1}{T}\int_T \left(\sum_{k=-\infty}^{\infty} C_x(k)e^{j\frac{2\pi kt}{T}}\right) x(t)dt \\
&= \sum_{k=-\infty}^{\infty} C_x(k)\cdot\frac{1}{T}\int_T x(t)e^{j\frac{2\pi kt}{T}}dt \\
&= \sum_{k=-\infty}^{\infty} C_x(k)\cdot C_x^*(k) = \sum_{k=-\infty}^{\infty} |C_x(k)|^2 = \sum_{k=-\infty}^{\infty} P_{xx}(k) = \text{RHS}
\end{aligned}$$

q.e.d.

Example 6.9

Find the energy of $x(t)$ discussed in example 6.7.

$$x(t) = e^{-at}u(t), \quad a > 0$$

Figure 6.25: $x(t)$.

Solution:

Example 6.10

Find the average power of the input $x(t)$ and the output $y(t)$ for the following LTI system, where the impulse response and the input signal are given respectively as:

$$h(t) = e^{-at}u(t), \quad a > 0$$

$$x(t) = \cos(t)$$

Figure 6.26: LTI system with $h(t)$ and $x(t)$.

Solution:

(a) Input power P_I:

We try to compute the average power in three different ways:

Figure 6.27: Power spectral density of the input: $P_{xx}(k)$.

(b) Output power P_O:

In order to compute the output power, we use the power spectral density of the output $P_{yy}(k)$ [1], and first we discuss the relationship between the input and the output in terms of power:

[1] In this way, we do need to compute the output signal $y(t)$ specifically, which simplifies much of the work required!!!

Relationship of input/output power spactral density

We have, for periodic signals $x(t)$ and $y(t)$ where $\omega_0 = \frac{2\pi}{T_0}$(rad/sec):

$$\begin{aligned} & |Y(\omega)|^2 = |H(\omega)|^2 \cdot |X(\omega)|^2 \\ \Longrightarrow \quad & |Y(k\omega_0)|^2 = |H(k\omega_0)|^2 \cdot |X(k\omega_0)|^2 \\ \Longrightarrow \quad & |2\pi C_y(k)|^2 = |H(k\omega_0)|^2 \cdot |2\pi C_x(k)|^2 \\ \Longrightarrow \quad & |C_y(k)|^2 = |H(k\omega_0)|^2 \cdot |C_x(k)|^2 \\ \Longrightarrow \quad & P_{yy}(k) = |H(k\omega_0)|^2 \cdot P_{xx}(k) \end{aligned}$$

e.g.: If $T_0 = 2\pi$(sec), then $\omega_0 = \frac{2\pi}{T_0} = 1$(rad/sec), and

$$P_{yy}(k) = |H(k)|^2 \cdot P_{xx}(k)$$

Note: From the figure of $P_{yy}(k)$, we can easily get the autocorrelation function of the output signal as:

$$R_{yy}(\tau) = \frac{1}{2(1+a^2)} \cos(\tau)$$

Figure 6.28: Power spectral density of the output, $P_{yy}(k)$, with $P_{xx}(k)$ and $|H(k)|^2$.