Contents

Chapter 7

DISCRETE LINEAR TIME INVARIANT(DLTI) SYSTEMS

7.1 Input/output relation in DLTI system: Convolution sum

Recall the definition of discrete-time signals, which are usually obtained by uniformly sampling the contiunuous-time signals, e.g.:

Figure 7.1: A discrete-time signal.

Discrete LTI system

$h[n]$ is the internal function representing the system characteristics

Figure 7.2: DLTI system

FACT:
The input/output signals of the DLTI system are related by a convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \triangleq x[n] * h[n] \tag{7.1}$$

where $h[n]$ is the *impulse response* of the system $T[\cdot]$.

System impulse response: $h[n]$

Figure 7.3: Concept of impulse response for a DLTI system

Definition 7.1 The impulse response $h[n]$ of a DLTI system is defined as the output signal when the input signal is the *unit sample function* [1] $\delta[n]$, i.e.,

$$h[n] \triangleq T[\delta[n]]$$

where the unit sample function is defined as follows:

$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Figure 7.4: Unit sample function $\delta[n]$

Check: validity of the definition of $h[n]$ using (7.1)

$$y[n] = T[\delta[n]] = \delta[n] * h[n] = \sum_{k=-\infty}^{\infty} \delta[k]h[n-k] = \delta[0] \cdot h[n-0] = h[n]$$

which is the impuse response of the DLTI system. Here, we used the following fact regarding the unit sample function:

$$\delta[n-k] \triangleq \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

[1] Recall the definition of the unit impulse function $\delta(t)$ for continuous case.

Brief Derivation of convolution sum(7.1)

A general discrete-time signal $x[n]$ is the sampled version of the continuous-time signal $x(t)$ with a uniform sampling period of T_s, and can be expressed as:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

: train(*linear combination*) of weighted and delayed unit sample functions where $x[k]$ is the sample value of $x(t)$ at $t = kT_s$, i.e. $x[k] = x(kT_s)$:

Figure 7.5: Representation of discrete $x[n]$

Due to the *linearity* and *time-invariance* properties of the DLTI system, the output signal $y[n]$ is expressed as:

$$\begin{aligned}
y[n] &= T\left[x[n]\right] \\
&= T\left[\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right] \\
&= \sum_{k=-\infty}^{\infty} x[k]\cdot T\left[\delta[n-k]\right] \quad \text{(linear system)} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{(time invariant system, and } h[n] \triangleq T\left[\delta[n]\right]\text{)} \\
&\triangleq x[n] * h[n] \qquad (7.2)
\end{aligned}$$

: **convolution sum**

Another expression of the convolution sum

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&\quad (Let\ n-k=m \quad \rightarrow \quad k=n-m) \\
&= \sum_{m=\infty}^{-\infty} x[n-m]h[m] \\
&= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\
&\triangleq h[n] * x[n] \qquad (7.3)
\end{aligned}
$$

Therefore, the output $y[n]$ of a DLTI system can be obtained by:

$$y[n] = h[n] * x[n] \overset{OR}{=} x[n] * h[n]$$

Note:
The choice between (7.2) and (7.3) to compute $y[n]$ is entirely depending on the easiness of calculation w.r.t. the associated $x[n]$ and $h[n]$!!!

Example 7.1

Find the output signal $y[n]$ of a DLTI system, when the input and the impulse response of the system are given repectively as follows:

$$x[n] = u[n] - u[n-N]$$

$$h[n] = a^n u[n], \quad \text{where } 0 < a < 1$$

where $u[n]$ is the discrete unit step function defined as follows:

$$u[n] \triangleq \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Figure 7.6: Input sequence $x[n]$, and the impulse response $h[n]$ of a DLTI system

Solution:

Compute the convolution sum between $x[n]$ and $h[n]$:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Figure 7.7: The convolution sum procedure.

Figure 7.8: Output sequence $y[n]$ of a DLTI system

Assignment: Try $y[n] = h[n] * x[n]$, and see if you get the same result!