Contents

Chapter 8

DISCRETE FOURIER SERIES

8.1 Concept of Discrete Fourier Series(DFS)

Basic Idea

Suppose we are given a *periodic*, *discrete* signal $x[n]$ with period of N, which is a uniformly sampled version of continuoius periodic signal $x(t)$, $\ni$:

$$x[n] = x[n + m \cdot N] \quad \text{where } m\text{:integer}$$

where N is called the *fundamental period* of $x[n]$, and it must be an *integer*.

e.g.:

Figure 8.1: Typical discrete periodic signal $x[n]$ w/ period N(samples).

Based upon the same reasoning behind the continuous Fourier series, we can express $x[n]$ as a linear combination of *harmonically* related discrete complex exponentials, where the fundamental frequency is now: [1]

$$\omega_0 = \frac{2\pi}{N} \text{ (rad)}$$

i.e.:

$$x[n] = \sum_{k=0}^{N-1} D_x(k) e^{j\frac{2\pi k}{N}n} \quad : \text{Discrete F.S.}$$

NOTE: The discrete Fourier series is a **finite series**, whereas the continuous Fourier series is an *infinite series* [2] !!!

[1]For continuous periodic signal $x(t) = x(t+T)$, the fundamental frequency is $\omega_0 = \frac{2\pi}{T}$(rad/sec).
[2]Recall that continuous F.S.: $x(t) = \sum_{k=-\infty}^{\infty} C_x(k) e^{j\frac{2\pi k}{T}t}$

8.2 Comparison between DFS and CFS

$$x(t) = \sum_{k=-\infty}^{\infty} C_x(k) e^{j\frac{2\pi k}{T}t} \quad : \text{CFS (infinite series)}$$

$$x[n] = \sum_{k=0}^{N-1} D_x(k) e^{j\frac{2\pi k}{N}n} \quad : \text{DFS (finite series)}$$

(1) Continuous Fourier Series(CFS):

Let

$$\phi_c(k) \triangleq e^{j\frac{2\pi k}{T}t} = \cos\left(\frac{2\pi kt}{T}\right) + j\sin\left(\frac{2\pi kt}{T}\right)$$

then, all of $\phi_c(k)$ are *distinct* for $k = -\infty$ to $k = \infty$, i.e. they never repeats!!!

Example 8.1

It is easy to notice that for $\phi_c(k)$, which is:

$$\phi_c(k) \triangleq e^{j\frac{2\pi k}{T}t} = \cos\left(\frac{2\pi kt}{T}\right) + j\sin\left(\frac{2\pi kt}{T}\right)$$

the real and imaginary parts, $\left\{\cos\left(\frac{2\pi kt}{T}\right)\right\}_{k=-\infty}^{\infty}$ and $\left\{\sin\left(\frac{2\pi kt}{T}\right)\right\}_{k=-\infty}^{\infty}$ respectively, are all distinct for different value of k, for instance, let $T = 2\pi$(sec), then the real and imaginary parts of $\phi_c(k)$:

(i) Real part:

$$\cos(t), \cos(2t), \ldots, \cos(kt), \ldots\ldots$$

(ii) Imaginary part:

$$\sin(t), \sin(2t), \ldots, \sin(kt), \ldots\ldots$$

are all different to each other.

NOTE:
This is why the CFS requires a infinite number of harmonic frequency components to completely represent a continuous periodic signal, and thus the CFS is in a infinite series form!!!

(2) Discrete Fourier Series(DFS):

Let

$$\phi_d(k) \triangleq e^{j\frac{2\pi k}{N}n} = \cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right)$$

then, $\phi_d(k)$ is periodic in k with period of N, i.e.:

$$\phi_d(k) = \phi_d(k + rN) \quad \text{where } r\text{: integer}$$

proof:

$$\begin{aligned} \text{RHS} &= \phi_d(k + rN) \\ &= e^{j\frac{2\pi(k+rN)}{N}n} \\ &= e^{j\frac{2\pi k}{N}n} \cdot e^{j\frac{2\pi rN}{N}n} \\ &= e^{j\frac{2\pi k}{N}n} \\ &= \phi_d(k) = \text{LHS} \end{aligned}$$

Example 8.2

Let the period $N = 4$, then:

$$\begin{aligned} \phi_d(k) \triangleq e^{j\frac{2\pi k}{N}n} &= \cos\left(\frac{2\pi kn}{N}\right) + j\sin\left(\frac{2\pi kn}{N}\right) \\ &= \cos\left(\frac{2\pi kn}{4}\right) + j\sin\left(\frac{2\pi kn}{4}\right) \\ &= \cos\left(\frac{\pi kn}{2}\right) + j\sin\left(\frac{\pi kn}{2}\right) \end{aligned}$$

The real and imaginary parts of $\phi_d(k)$, $\left\{\cos\left(\frac{\pi kn}{2}\right)\right\}_{k=-\infty}^{\infty}$ and $\left\{\sin\left(\frac{\pi kn}{2}\right)\right\}_{k=-\infty}^{\infty}$ respectively, repeat themselves in k with the period of $N = 4$ as follows:

(i) Real part:

$$\begin{aligned}
&\cos\left(\frac{\pi}{2}n\right), \cos(\pi n), \cos\left(\frac{3\pi}{2}n\right), \cos(2\pi n), \quad k = 1,2,3,4 \\
&\cos\left(\frac{5\pi}{2}n\right), \cos(3\pi n), \cos\left(\frac{7\pi}{2}n\right), \cos(4\pi n), \quad k = 5,6,7,8 \\
&\left(\equiv \cos\left(\frac{\pi}{2}n\right), \cos(\pi n), \cos\left(\frac{3\pi}{2}n\right), \cos(2\pi n)\right) \\
&\vdots \\
&\vdots
\end{aligned}$$

(ii) Imaginary part:

$$\begin{aligned}
&\sin\left(\frac{\pi}{2}n\right), \sin(\pi n), \sin\left(\frac{3\pi}{2}n\right), \sin(2\pi n), \quad k = 1,2,3,4 \\
&\sin\left(\frac{5\pi}{2}n\right), \sin(3\pi n), \sin\left(\frac{7\pi}{2}n\right), \sin(4\pi n), \quad k = 5,6,7,8 \\
&\left(\equiv \sin\left(\frac{\pi}{2}n\right), \sin(\pi n), \sin\left(\frac{3\pi}{2}n\right), \sin(2\pi n)\right) \\
&\vdots \\
&\vdots
\end{aligned}$$

NOTE:
This is why the DFS requires **only** a finite number(N) of harmonic frequency components, from d.c. to $\frac{2\pi(N-1)}{N}$(rad), to completely represent a discrete periodic signal, and thus the DFS is in a finite series form!!!

8.3 Representation of Complex DFS

Let $x[n]$ be a *periodic*, *discrete* signal with period of N, i.e.

$$x[n] = x[n + m \cdot N] \quad \text{where } m\text{:integer}$$

where N is called the *fundamental period* of $x[n]$, and it must be an *integer*.

Then, based upon the same reasoning behind the continuous Fourier series, we can express $x[n]$ as a linear combination of *harmonically* related discrete complex exponentials,

$$x[n] = \sum_{k=0}^{N-1} D_x(k) e^{j\frac{2\pi k}{N}n}$$

where the fundamental frequency is $\omega_0 = \frac{2\pi}{N}$ (rad).

The corresponding DFS coefficients [3] $\{D_x(k)\}_{k=-\infty}^{\infty}$ are given:

$$D_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

Note:

1. In general, $D_x(k)$'s are complex numbers, i.e.:

$$\begin{aligned} D_x(k) &= \text{Re}[D_x(k)] + j\text{Im}[D_x(k)] \quad \text{(cartesian coordinate)} \\ &= |D_x(k)| e^{j\Phi_x(k)} \quad \text{(polar coordinate)} \quad : \textit{preferred}!!! \end{aligned}$$

   where

$$|D_x(k)| = \sqrt{\text{Re}^2[D_x(k)] + \text{Im}^2[D_x(k)]}$$

$$\Phi_x(k) = \arctan\left\{\frac{\text{Im}[D_x(k)]}{\text{Re}[D_x(k)]}\right\}$$

2. Since $\phi_d(k)$ is periodic, $D_x(k)$, naturally, would be periodic in k with period of N: to be officially proved later.

[3] Recall that for continuous F.S., the coefficient is in the form of $C_x(k) = \frac{1}{T}\int_T x(t) e^{-j\frac{2\pi k t}{T}} dt$.

Derivation of DFS coefficient: $D_x(k)$

FACT:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} = \begin{cases} N, & k = m \cdot N \\ 0 & \text{otherwise} \end{cases}$$

proof:

(i) $k \neq m \cdot N$:

$$\begin{aligned} \sum_{n=0}^{N-1} \left(e^{j\frac{2\pi k}{N}}\right)^n &= \frac{1 - e^{j\frac{2\pi k}{N}N}}{1 - e^{j\frac{2\pi k}{N}}} \\ &= \frac{1 - e^{j2\pi k}}{1 - e^{j\frac{2\pi k}{N}}} \\ &= 0 \end{aligned}$$

(ii) $k = m \cdot N$:

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi mNn}{N}} = \sum_{n=0}^{N-1} 1 = N$$

q.e.d.

Now, to derive the DFS coefficient $D_x(k)$, from

$$x[n] = \sum_{k=0}^{N-1} D_x(k) e^{j\frac{2\pi k}{N}n}$$

Multiply both sides with $e^{-j\frac{2\pi rn}{N}}$, where r is integers ranging $0 \leq r \leq N-1$, and take summation $\sum_{n=0}^{N-1}$, then

$$\begin{aligned} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi rn}{N}} &= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} D_x(k) e^{j\frac{2\pi(k-r)n}{N}} \\ &= \sum_{k=0}^{N-1} D_x(k) \sum_{n=0}^{N-1} e^{j\frac{2\pi(k-r)n}{N}} \end{aligned} \tag{8.1}$$

In the above equation (8.1), notice from the **FACT** that

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi(k-r)n}{N}} = \begin{cases} N, & k - r = m \cdot N \rightarrow k = r + mN \rightarrow k = r \\ 0 & k - r \neq m \cdot N \rightarrow k \neq r + mN \rightarrow k \neq r \end{cases}$$

where the value of k is restrained as $0 \leq k \leq N-1$. From (8.1), we have

$$\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi rn}{N}} = N \cdot D_x(r)$$

Therefore, the discrete Fourier series coefficient $D_x(k)$ can be put into the following formula:

$$D_x(r) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi rn}{N}}$$

or replacing r with k, we get:

$$D_x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

In summary, we have the following DFS pair relationship for periodic discrete-time signals $x[n]$;

$$x[n] = \sum_{k=0}^{N-1} D_x(k)e^{j\frac{2\pi k}{N}n}$$

$$D_x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

Note: $D_x(k)$ is periodic in k with period of N.

proof:

$$\begin{aligned}
D_x(k+mN) &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi(k+mN)}{N}n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \cdot e^{-j\frac{2\pi mN}{N}n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \\
&\triangleq D_x(k)
\end{aligned}$$