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Chapter 9

DISCRETE TIME FOURIER TRANSFORM

9.1 Concept of Discrete Time Fourier Transform (DTFT)

Basic Idea

We are given a non-periodic discrete-time signal $x[n]$ defined for all $-\infty < n < \infty$, where n is *integer*:

e.g.:

Figure 9.1: Typical non-periodic discrete signal $x[n]$.

Note: Notice that $x[n]$ samples are almost zero, for outside of $0 \leq n \leq N-1$, i.e.

$$x[n] \approx 0, \quad \text{for } n \geq N \text{ and } n < 0$$

Similarly to the procedure to derive F.T. from F.S. for non-periodic continuous signal, we defined a *discrete periodic* signal $\tilde{x}[n]$ from $x[n]$ as follows:

$$\tilde{x}[n] = x[n], \quad 0 \leq n < N \qquad \text{:truncation}$$

and

$$\tilde{x}[n] = \tilde{x}[n + m \cdot N]$$

where m is an integer.

Figure 9.2: Periodic discrete signal $\tilde{x}[n]$ constructed from non-peridic $x[n]$.

Then, we can express $\tilde{x}[n]$ as a discrete Fourier series(DFS), and by letting $N \to \infty$, we get the analysis tool (DTFT) of $x[n]$ in the frequency domain, since,

$$\lim_{N \to \infty} \tilde{x}[n] = x[n]$$

9.2 Representation of DTFT

Let $x[n]$ be a non-periodic discrete-time signal of which sample values are almost zero for outside of $0 \leq n \leq N-1$, and construct a discrete periodic signal $\tilde{x}[n]$ from $x[n]$ as discussed in the previous section, i.e.:

$$\tilde{x}[n] = x[n], \quad 0 \leq n < N$$

and

$$\tilde{x}[n] = \tilde{x}[n + m \cdot N]$$

Then, the discrete Fourier series(DFS) pair of $\tilde{x}[n]$ is as follows:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{D}_x(k) e^{j\frac{2\pi kn}{N}}$$

$$\tilde{D}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}}$$

Notice that

$$\begin{aligned}
\tilde{D}_x(k) &= \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \\
&\quad (\text{since } \tilde{x}[n] = x[n], \quad 0 \leq n \leq N-1) \\
&= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi kn}{N}} \\
&\quad (\text{since } x[n] \approx 0, \quad \text{for } n < 0,\ n \geq N)
\end{aligned}$$

Note:
In this form itself, $\tilde{D}_x(k) \to 0$ as $N \to \infty$, and thus meaningless!!! i.e.

$$\lim_{N \to \infty} \tilde{D}_x(k) = 0$$

So, we define the following function in order to represent the frequency distribution of a non-periodic discrete signal $x[n]$, which is:

$$X\left(e^{j\omega}\right) \triangleq \lim_{N\to\infty} N \cdot \tilde{D}_x(k) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad : \text{DTFT of } x[n] \tag{9.1}$$

where ω is a *continuous* variable which comes from the asymptotic equation below:

$$\lim_{N\to\infty} \frac{2\pi k}{N} \longrightarrow \omega$$

Remark: The discrete harmonic frequncies $\left\{\frac{2\pi k}{N}\right\}_{k=0}^{N-1}$ become *continuous*!!!

Then, the DFS coefficient $\tilde{D}_x(k)$ can be expressed as:

$$\begin{aligned}
\tilde{D}_x(k) &= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \frac{1}{N}\, X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi k}{N}\triangleq k\omega_0} \\
&= \frac{1}{N} X\left(e^{jk\omega_0}\right) \\
& \quad (\text{where } \omega_0 \triangleq \tfrac{2\pi}{N}, \text{ and thus } N = \tfrac{2\pi}{\omega_0}) \\
&= \frac{\omega_0}{2\pi} X\left(e^{jk\omega_0}\right)
\end{aligned}$$

which is the uniformly sampled version of $\frac{1}{N}X\left(e^{j\omega}\right)$ with spacing of ω_0.

Relation between $X\left(e^{j\omega}\right)$ and $\tilde{D}_x(k)$:

Note that $X\left(e^{j\omega}\right)$ is the *envelope* of $N \cdot \tilde{D}_x(k)$, where discrete harmonic frequencies $\left\{\frac{2\pi k}{N} = k\omega_0\right\}_{k=0}^{N-1}$ become continuous ω !!!,
i.e.

Figure 9.3: Relation between $X\left(e^{j\omega}\right)$ and $\tilde{D}_x(k)$ for one period.

Now, from the DFS pair of $\tilde{x}[n]$, we have:

$$\begin{aligned}
\tilde{x}[n] &= \sum_{k=0}^{N-1} \tilde{D}_x(k) e^{j\frac{2\pi kn}{N}} \\
&= \sum_{k=0}^{N-1} \frac{1}{N} X\left(e^{jk\omega_0}\right) \cdot e^{jk\omega_0 n} \\
&= \frac{1}{2\pi} \sum_{k=0}^{N-1} X\left(e^{jk\omega_0}\right) \cdot e^{jk\omega_0 n} \cdot \omega_0 \qquad (9.2)
\end{aligned}$$

As $N \to \infty$, we have the follwoing facts:

$$\begin{aligned}
&(i) \;\; \tilde{x}[n] \longrightarrow x[n] \\
&(ii) \;\; \omega_0 = \frac{2\pi}{N} \longrightarrow d\omega \\
&(iii) \;\; k\omega_0 = \frac{2\pi k}{N} \longrightarrow \omega \\
&(iv) \;\; \sum_{k=0}^{N-1} \longrightarrow \int_0^{2\pi} \\
&(\text{since } \tfrac{(N-1)2\pi}{N} \;\; \rightarrow \;\; 2\pi \text{ as } N \rightarrow \infty)
\end{aligned}$$

Applying *(i)* to *(iv)* into (9.2), we have:

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X\left(e^{j\omega}\right) \cdot e^{j\omega n} d\omega \qquad (9.3)$$

From (9.1) and (9.3), we have the following DTFT pair for non-periodic discrete signal $x[n]$:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad : \text{DTFT}$$

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X\left(e^{j\omega}\right) \cdot e^{j\omega n} d\omega \qquad : \text{IDTFT}$$

Remarks:

1. $X(e^{j\omega})$ is a continuous function of ω even though $x[n]$ is a discrete signal! [1]

2. $X(e^{j\omega})$ is periodic in ω with period of 2π(rad), i.e.

$$X\left(e^{j\omega}\right) = X\left(e^{j(\omega+2\pi m)}\right)$$

proof:

$$\begin{aligned} \text{RHS} &= \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j(\omega+2\pi m)n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \cdot e^{-j2\pi mn} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ &\triangleq X\left(e^{j\omega}\right) = \text{LHS} \end{aligned}$$

(cf) We can deduce this fact that: since $X(e^{j\omega})$ is the envelope of $N \cdot \tilde{D}_x(k)$, and $N \cdot \tilde{D}_x(k)$ is periodic in k with period of N, $X(e^{j\omega})$ must also be periodic!

Example 9.1

Determine the frequency distribution of a non-periodic signal $x[n]$ given below $\ni$:

$$x[n] = a^n u[n], \qquad |a| < 1 \text{ and } a > 0$$

Figure 9.4: A non-periodic $x[n]$.

[1]This is due to the "non-periodicity" of $x[n]$, and will be discussed near the end of the semester.

Solution:

Figure 9.5: The magnitude and the phase spectra.

Note: Check out the fact that $X\left(e^{j\omega}\right)$ is periodic in ω with period of 2π(radian).

9.3 Properties of DTFT

The properties of DTFT are similar to the properties of F.T. for continuous non-periodic signals $x(t)$ discussed in chapter 4, except for few cases:

Let

$$x[n] \stackrel{\mathrm{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)$$

then, some typical properties of the DTFT are as follows:

(1) Linearity

$$\begin{aligned}\mathrm{F}\left[\sum_{i=1}^{N} a_i x_i[n]\right] &= \sum_{i=1}^{N} a_i \mathrm{F}\left[x_i[n]\right] \\ &= \sum_{i=1}^{N} a_i X_i\left(e^{j\omega}\right)\end{aligned}$$

(2) Time shift

$$\mathrm{F}\left[x[n-n_0]\right] = e^{-j\omega n_0} \cdot X\left(e^{j\omega}\right)$$

(3) Frequency shift

$$\mathrm{F}\left[e^{j\omega_0 n} \cdot x[n]\right] = X\left(e^{j(\omega-\omega_0)}\right)$$

(4) Time scaling

Define a time scaled discrete signal $x_k[n]$ as:

$$x_k[n] \triangleq \begin{cases} x[\frac{n}{k}] & \text{if } n = m \cdot k \\ \\ 0 & \text{if } n \neq m \cdot k \end{cases}$$

Then,

$$\text{F}\left[x_k[n]\right] = X\left(e^{jk\omega}\right)$$

Note: Comparison with the continuous F.T.:

$$\mathcal{F}[x(at)] = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

1. $x[an]$, where a is real, is NOT proper in this case, since an cannot be an integer!
2. $x[kn]$, where k is an integer, is merely a *resampling* of $x[n]$,
   e.g.

(NOT either compression or stretching) (Stretched version of $x[n]$)

Figure 9.6: Resampled $x[2n]$ and twice stretched $x[\frac{n}{2}]$ from $x[n]$.

(5) Differentiation in frequency

$$\mathrm{F}\left[nx[n]\right] = j\frac{dX\left(e^{j\omega}\right)}{d\omega}$$

Note: We cannot consider the differentiation and/or the integration of $x[n]$ since it is a discrete signal. Instead, we should consider the *difference* and *summation*: refer the Table5.1 of the main reference book.

(6) Convolution

$$\mathrm{F}\left[x[n] * h[n]\right] = X\left(e^{j\omega}\right) \cdot H\left(e^{j\omega}\right)$$

Note: Input/output relationship of a DLTI system.

(7) Modulation

$$\mathrm{F}\left[x[n] \cdot y[n]\right] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\Omega}\right) Y\left(e^{j(\omega-\Omega)}\right) d\Omega$$

: Periodic Convolution

Reminder: $X\left(e^{j\omega}\right)$ is periodic with period 2π(rad).

PROOF of (1) $\sim$ (7): Assignment